Example 7.11 Show that in the free oscillations of an LC circuit, the sum of energies stored in the capacitor and the inductor is constant in time.

Solution Let q_0 be the initial charge on a capacitor. Let the charged capacitor be connected to an inductor of inductance L. As you have studied in Section 7.8, this LC circuit will sustain an oscillation with frquency

$$\omega \left(= 2\pi\nu = \frac{1}{\sqrt{LC}} \right)$$

At an instant t, charge q on the capacitor and the current i are given by:

$q(t) = q_0 \cos \omega t$

$i(t) = -q_0 \, \omega \sin \omega t$

Energy stored in the capacitor at time t is

$$U_E = \frac{1}{2} C\, V^2 = \frac{1}{2}\frac{q^2}{C} = \frac{q_0^2}{2C} \cos^2(\omega t)$$

Energy stored in the inductor at time t is

$$U_M = \frac{1}{2} L\, i^2$$

$$= \frac{1}{2} L\, q_0^2\, \omega^2 \sin^2(\omega t)$$

$$= \frac{q_0^2}{2C} \sin^2(\omega t) \qquad (\because \omega = 1/\sqrt{LC})$$

Sum of energies

$$U_E + U_M = \frac{q_0^2}{2C}\left(\cos^2 \omega t + \sin^2 \omega t\right)$$

$$= \frac{q_0^2}{2C}$$

This sum is constant in time as q_o and C, both are time-independent. Note that it is equal to the initial energy of the capacitor. Why it is so? Think!

EXAMPLE 7.11

7.9 TRANSFORMERS

For many purposes, it is necessary to change (or transform) an alternating voltage from one to another of greater or smaller value. This is done with a device called *transformer* using the principle of mutual induction.

A transformer consists of two sets of coils, insulated from each other. They are wound on a soft-iron core, either one on top of the other as in Fig. 7.20(a) or on separate limbs of the core as in Fig. 7.20(b). One of the coils called the *primary coil* has N_p turns. The other coil is called the *secondary coil*; it has N_s turns. Often the primary coil is the input coil and the secondary coil is the output coil of the transformer.

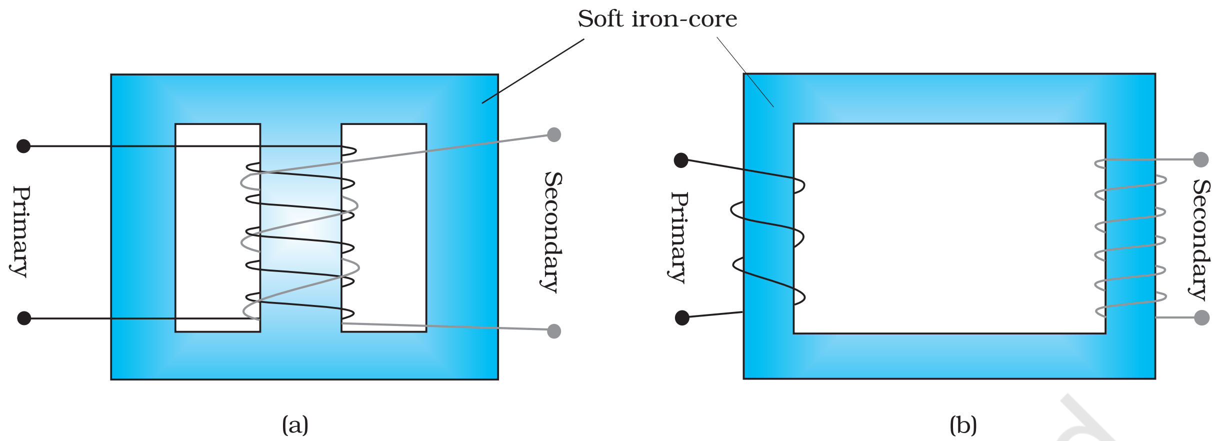

FIGURE 7.20 Two arrangements for winding of primary and secondary coil in a transformer: (a) two coils on top of each other, (b) two coils on separate limbs of the core.

When an alternating voltage is applied to the primary, the resulting current produces an alternating magnetic flux which links the secondary and induces an emf in it. The value of this emf depends on the number of turns in the secondary. We consider an ideal transformer in which the primary has negligible resistance and all the flux in the core links both primary and secondary windings. Let ϕ be the flux in each turn in the core at time t due to current in the primary when a voltage v_p is applied to it.

Then the induced emf or voltage ε_s, in the secondary with N_s turns is

$$\varepsilon_s = -N_s \frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{7.45}$$

The alternating flux ϕ also induces an emf, called back emf in the primary. This is

$$\varepsilon_p = -N_p \frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{7.46}$$

But $\varepsilon_p = v_p$. If this were not so, the primary current would be infinite since the primary has zero resistance (as assumed). If the secondary is an open circuit or the current taken from it is small, then to a good approximation

$$\varepsilon_s = v_s$$

where v_s is the voltage across the secondary. Therefore, Eqs. (7.45) and (7.46) can be written as

$$v_s = -N_s \frac{d\phi}{dt} \tag{7.45(a)}$$

$$v_p = -N_p \frac{d\phi}{dt} \tag{7.46(a)}$$

From Eqs. [7.45 (a)] and [7.46 (a)], we have

$$\frac{v_s}{v_p} = \frac{N_s}{N_p} \tag{7.47}$$

Note that the above relation has been obtained using three assumptions: (i) the primary resistance and current are small; (ii) the same flux links both the primary and the secondary as very little flux escapes from the core, and (iii) the secondary current is small.

If the transformer is assumed to be 100% efficient (no energy losses), the power input is equal to the power output, and since $p = i\,v$,

$$i_p v_p = i_s v_s \tag{7.48}$$

Although some energy is always lost, this is a good approximation, since a well designed transformer may have an efficiency of more than 95%. Combining Eqs. (7.47) and (7.48), we have

$$\frac{i_p}{i_s} = \frac{v_s}{v_p} = \frac{N_s}{N_p} \tag{7.49}$$

Since i and v both oscillate with the same frequency as the ac source, Eq. (7.49) also gives the ratio of the amplitudes or rms values of corresponding quantities.

Now, we can see how a transformer affects the voltage and current. We have:

$$V_s = \left(\frac{N_s}{N_p}\right) V_p \quad \text{and} \quad I_s = \left(\frac{N_p}{N_s}\right) I_p \tag{7.50}$$

That is, if the secondary coil has a greater number of turns than the primary ($N_s > N_p$), the voltage is stepped up ($V_s > V_p$). This type of arrangement is called a *step-up transformer*. However, in this arrangement, there is less current in the secondary than in the primary ($N_p/N_s < 1$ and $I_s < I_p$). For example, if the primary coil of a transformer has 100 turns and the secondary has 200 turns, $N_s/N_p = 2$ and $N_p/N_s = 1/2$. Thus, a 220V input at 10A will step-up to 440 V output at 5.0 A.

If the secondary coil has less turns than the primary ($N_s < N_p$), we have a *step-down transformer*. In this case, $V_s < V_p$ and $I_s > I_p$. That is, the voltage is stepped down, or reduced, and the current is increased.

The equations obtained above apply to ideal transformers (without any energy losses). But in actual transformers, small energy losses do occur due to the following reasons:

(i) *Flux Leakage*: There is always some flux leakage; that is, not all of the flux due to primary passes through the secondary due to poor design of the core or the air gaps in the core. It can be reduced by winding the primary and secondary coils one over the other.

(ii) *Resistance of the windings*: The wire used for the windings has some resistance and so, energy is lost due to heat produced in the wire (I^2R). In high current, low voltage windings, these are minimised by using thick wire.

(iii) *Eddy currents*: The alternating magnetic flux induces eddy currents in the iron core and causes heating. The effect is reduced by using a laminated core.

(iv) *Hysteresis*: The magnetisation of the core is repeatedly reversed by the alternating magnetic field. The resulting expenditure of energy in the core appears as heat and is kept to a minimum by using a magnetic material which has a low hysteresis loss.

The large scale transmission and distribution of electrical energy over long distances is done with the use of transformers. The voltage output of the generator is stepped-up (so that current is reduced and consequently, the I^2R loss is cut down). It is then transmitted over long distances to an area sub-station near the consumers. There the voltage is stepped down. It is further stepped down at distributing sub-stations and utility poles before a power supply of 240 V reaches our homes.

SUMMARY

1. An alternating voltage $v = v_m \sin \omega t$ applied to a resistor R drives a current $i = i_m \sin\omega t$ in the resistor, $i_m = \dfrac{v_m}{R}$. The current is in phase with the applied voltage.
2. For an alternating current $i = i_m \sin \omega t$ passing through a resistor R, the average power loss P (averaged over a cycle) due to joule heating is $(1/2)i_m^2R$. To express it in the same form as the dc power ($P = I^2R$), a special value of current is used. It is called *root mean square* (rms) *current* and is donoted by I:

$$I = \frac{i_m}{\sqrt{2}} = 0.707\, i_m$$

Similarly, the *rms voltage* is defined by

$$V = \frac{v_m}{\sqrt{2}} = 0.707\, v_m$$

We have $P = IV = I^2R$

3. An ac voltage $v = v_m \sin \omega t$ applied to a pure inductor L, drives a current in the inductor $i = i_m \sin (\omega t - \pi/2)$, where $i_m = v_m/X_L$. $X_L = \omega L$ is called *inductive reactance*. The current in the inductor lags the voltage by $\pi/2$. The average power supplied to an inductor over one complete cycle is zero.
4. An ac voltage $v = v_m \sin\omega t$ applied to a capacitor drives a current in the capacitor: $i = i_m \sin (\omega t + \pi/2)$. Here,

$$i_m = \frac{v_m}{X_C},\ X_C = \frac{1}{\omega C}$$ is called *capacitive reactance.*

The current through the capacitor is $\pi/2$ ahead of the applied voltage. As in the case of inductor, the average power supplied to a capacitor over one complete cycle is zero.

5. For a series *RLC* circuit driven by voltage $v = v_m \sin \omega t$, the current is given by $i = i_m \sin (\omega t + \phi)$

where $$i_m = \frac{v_m}{\sqrt{R^2 + (X_C - X_L)^2}}$$

and $$\phi = \tan^{-1}\frac{X_C - X_L}{R}$$

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$ is called the *impedance* of the circuit.

The average power loss over a complete cycle is given by

$$P = V\,I\cos\phi$$

The term $\cos\phi$ is called the *power factor*.

6. In a purely inductive or capacitive circuit, $\cos\phi = 0$ and no power is dissipated even though a current is flowing in the circuit. In such cases, current is referred to as a *wattless current*.
7. The phase relationship between current and voltage in an ac circuit can be shown conveniently by representing voltage and current by rotating vectors called *phasors*. A phasor is a vector which rotates about the origin with angular speed ω. The magnitude of a phasor represents the amplitude or peak value of the quantity (voltage or current) represented by the phasor.

   The analysis of an ac circuit is facilitated by the use of a phasor diagram.
8. An interesting characteristic of a series RLC circuit is the phenomenon of *resonance*. The circuit exhibits resonance, i.e., the amplitude of the current is maximum at the resonant frequency, $\omega_0 = \dfrac{1}{\sqrt{LC}}$. The *quality factor* Q defined by

   $Q = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 CR}$ is an indicator of the sharpness of the resonance, the higher value of Q indicating sharper peak in the current.
9. A circuit containing an inductor L and a capacitor C (initially charged) with no ac source and no resistors exhibits *free oscillations*. The charge q of the capacitor satisfies the equation of simple harmonic motion:

   $$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

   and therefore, the frequency ω of free oscillation is $\omega_0 = \dfrac{1}{\sqrt{LC}}$. The energy in the system oscillates between the capacitor and the inductor but their sum or the total energy is constant in time.
10. A transformer consists of an iron core on which are bound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected to an ac source, the primary and secondary voltages are related by

    $$V_s = \left(\frac{N_s}{N_p}\right)V_p$$

    and the currents are related by

    $$I_s = \left(\frac{N_p}{N_s}\right)I_p$$

    If the secondary coil has a greater number of turns than the primary, the voltage is stepped-up ($V_s > V_p$). This type of arrangement is called a *step-up transformer*. If the secondary coil has turns less than the primary, we have a *step-down transformer*.

Physical quantity	Symbol	Dimensions	Unit	Remarks
rms voltage	V	$[\mathrm{M\,L^2T^{-3}A^{-1}}]$	V	$V = \dfrac{v_m}{\sqrt{2}}$, v_m is the amplitude of the ac voltage.
rms current	I	[A]	A	$I = \dfrac{i_m}{\sqrt{2}}$, i_m is the amplitude of the ac current.
Reactance:				
Inductive	X_L	$[\mathrm{M\,L^2T^{-3}A^{-2}}]$		$X_L = \omega L$
Capacitive	X_C	$[\mathrm{M\,L^2T^{-3}A^{-2}}]$		$X_C = 1/\omega C$
Impedance	Z	$[\mathrm{M\,L^2T^{-3}A^{-2}}]$		Depends on elements present in the circuit.
Resonant frequency	ω_r or ω_0	$[\mathrm{T^{-1}}]$	Hz	$\omega_0 = \dfrac{1}{\sqrt{LC}}$ for a series *RLC* circuit
Quality factor	Q	Dimensionless		$Q = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 C R}$ for a series *RLC* circuit.
Power factor		Dimensionless		$= \cos\phi$, ϕ is the phase difference between voltage applied and current in the circuit.

POINTS TO PONDER

1. When a value is given for ac voltage or current, it is ordinarily the rms value. The voltage across the terminals of an outlet in your room is normally 240 V. This refers to the *rms* value of the voltage. The amplitude of this voltage is

$$v_m = \sqrt{2}V = \sqrt{2}(240) = 340\ \text{V}$$

2. The power rating of an element used in ac circuits refers to its average power rating.
3. The power consumed in an ac circuit is never negative.
4. Both alternating current and direct current are measured in amperes. But how is the ampere defined for an alternating current? It cannot be derived from the mutual attraction of two parallel wires carrying ac currents, as the dc ampere is derived. An ac current changes direction

with the source frequency and the attractive force would average to zero. Thus, the ac ampere must be defined in terms of some property that is independent of the direction of the current. Joule heating is such a property, and there is one ampere of *rms* value of alternating current in a circuit if the current produces the same average heating effect as one ampere of dc current would produce under the same conditions.

5. In an ac circuit, while adding voltages across different elements, one should take care of their phases properly. For example, if V_R and V_C are voltages across R and C, respectively in an RC circuit, then the total voltage across RC combination is $V_{RC} = \sqrt{V_R^2 + V_C^2}$ and not $V_R + V_C$ since V_C is $\pi/2$ out of phase of V_R.
6. Though in a phasor diagram, voltage and current are represented by vectors, these quantities are not really vectors themselves. They are scalar quantities. It so happens that the amplitudes and phases of harmonically varying scalars combine mathematically in the same way as do the projections of rotating vectors of corresponding magnitudes and directions. The 'rotating vectors' that represent harmonically varying scalar quantities are introduced only to provide us with a simple way of adding these quantities using a rule that we already know as the law of vector addition.
7. There are no power losses associated with pure capacitances and pure inductances in an ac circuit. The only element that dissipates energy in an ac circuit is the resistive element.
8. In a RLC circuit, resonance phenomenon occur when $X_L = X_C$ or $\omega_0 = \dfrac{1}{\sqrt{LC}}$. For resonance to occur, the presence of both L and C elements in the circuit is a must. With only one of these (L or C) elements, there is no possibility of voltage cancellation and hence, no resonance is possible.
9. The power factor in a RLC circuit is a measure of how close the circuit is to expending the maximum power.
10. In generators and motors, the roles of input and output are reversed. In a motor, electric energy is the input and mechanical energy is the output. In a generator, mechanical energy is the input and electric energy is the output. Both devices simply transform energy from one form to another.
11. A transformer (step-up) changes a low-voltage into a high-voltage. This does not violate the law of conservation of energy. The current is reduced by the same proportion.
12. The choice of whether the description of an oscillatory motion is by means of sines or cosines or by their linear combinations is unimportant, since changing the zero-time position transforms the one to the other.