Find the coefficient of x^r in $(1-x)^{-n}$

Solution: -

We know that,

$\cancel{(1-x)^{-n} = 1-(x)(-n)}$

$$(1-x)^{-n} = 1 + (-n)(-x) + (-x)^2 \frac{(-n)(-n-1)}{2!} + (-x^3)\frac{(-n)(-n-1)(-n-2)}{3!}$$

$$+ \dots + (-x)^r \frac{(-n)(-n-1)(-n-2)\dots(-n-(r-1))}{r!} + \dots$$

$$= 1 + nx + \frac{x^2}{2!}(n)(n+1) + x^3 \frac{(n)(n+1)(n+2)}{3!} +$$

$$+ \frac{x^r}{r!}(n)(n+1)(n+2)\dots(n+r-1) + \dots\dots$$

$$= 1 + {}^nC_1\, x + x^2\, {}^{n+1}C_2 + x^3\, {}^{n+2}C_3 + \dots + x^r\, {}^{n+r-1}C_r \dots$$

If you observe the terms,

i.e., $\frac{(n)(n+1)(n+2)(n+3)\dots(n+r-1)}{r!}$

multiply and divide $(n-1)!$

$$\frac{(n)(n+1)(n+2)\dots(n+r-1)\times(n-1)!}{r!\,(n-1)!}$$

$$= \frac{[1\times 2\times 3\times\dots\times(n-1)]\,(n)(n+1)(n+2)\dots(n+r-1)}{r!\,(n-1)!}$$

$$= \frac{(n+r-1)!}{r!\,(n-1)!} = {}^{n+r-1}C_r$$

hence proved

$\therefore$ Coefficient of x^r is ${}^{n+r-1}C_r$, $n>0$, $r>0$