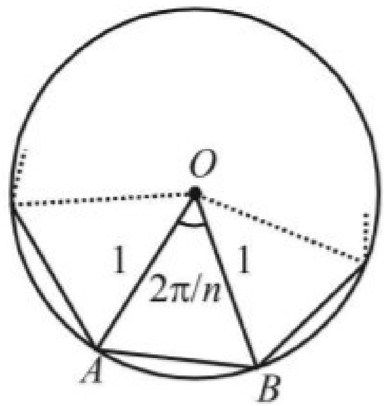
3 . If I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right).$$ **(2003 - 4 Marks)**

Solution: -

3 . Let OAB be one triangle out of n of a n sided polygon inscribed in a circle of radius 1.



Then $\angle AOB = \frac{2\pi}{n}$

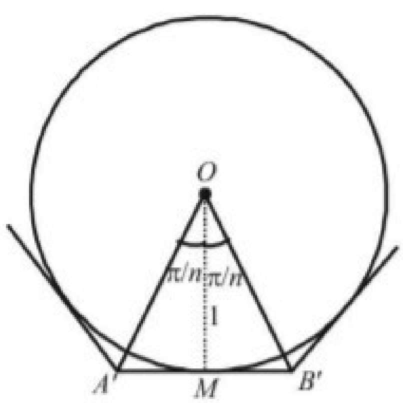
$OA = OB = 1$

∴ Using Area of isosceles Δ with vertical ∠θ and equal sides as

$r = \frac{1}{2}r^2 \sin\theta = \frac{1}{2}\sin\frac{2\pi}{n}$

∴ $I_n = \frac{n}{2}\sin\frac{2\pi}{n}$ ...... (1)

Further consider the n sided polygon subscribing on the circle.



$A'MB'$ is the tangent of the circle at M.

⇒ $A'MB' \perp OM$

⇒ $A'MO$ is right angled triangle, right angle at M.

$A'M = \tan\frac{\pi}{n}$

So, $O_n = n \tan \frac{\pi}{n}$ ....(2)

Now, we have to prove

$$I_n = \frac{O_n}{2}\left(1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}\right) \quad \text{or} \quad \frac{2I_n}{O_n} - 1 = \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}$$

$$\text{LHS} = \frac{2I_n}{O_n} - 1 = \frac{n \sin \frac{2\pi}{n}}{n \tan \frac{\pi}{n}} - 1 \text{ (From (1) and (2))}$$

$$= 2\cos^2 \frac{\pi}{n} - 1 = \cos \frac{2\pi}{n}$$

$$\text{RHS} = \sqrt{1 - \left(\frac{2I_n}{n}\right)^2} = \sqrt{1 - \sin^2 \frac{2\pi}{n}} \text{ (From (1))}$$

$$= \cos\left(\frac{2\pi}{n}\right)$$

Hence Proved.