5. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1: 2 is a parabola. Find the vertex of this parabola. ***(1995 - 5 Marks)***

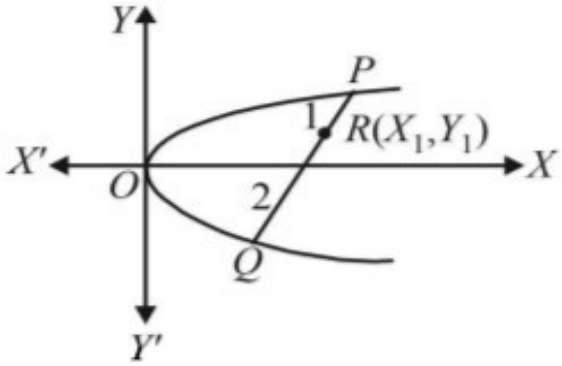
Solution: -

5. Let $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ be the ends of the chord PQ of the parabola

$$y^2 = 4x \qquad ...(1)$$

$$\therefore \quad \text{Slope of chord } PQ = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = 2$$

$$\Rightarrow \quad t_2 + t_1 = 1 \qquad ...(2)$$



If $R(x_1, y_1)$ is a point dividing PQ internally in the ratio $1 : 2$, then

$$x_1 = \frac{1.t_2^2 + 2.t_1^2}{1+2}, \quad y_1 = \frac{1.2t_2 + 2.2t_1}{1+2}$$

$$\Rightarrow \quad t_2^2 + 2t_1^2 = 3x_1 \qquad ...(3)$$

$$\text{and} \quad t_2 + 2t_1 = (3y_1)/2 \qquad ...(4)$$

From (2) and (4), we get

$$t_1 = \frac{3}{2}y_1 - 1, t_2 = 2 - \frac{3}{2}y_1$$

Substituting in (3), we get

$$\left(2 - \frac{3}{2}y_1\right)^2 + 2\left(\frac{3}{2}y_1 - 1\right)^2 = 3x_1$$

$\Rightarrow \quad (9/4)y_1^2 - 4y_1 = x_1 - 2$

$$\left(y_1 - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x_1 - \frac{2}{9}\right)$$

$\therefore$ Locus of the point $R(x_1, y_1)$ is

$(y - 8/9)^2 = (4/9)\,(x - 2/9)$

which is a parabola having vertex at the point (2/9, 8/9).