4. Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ.

(1994 - 4 Marks)

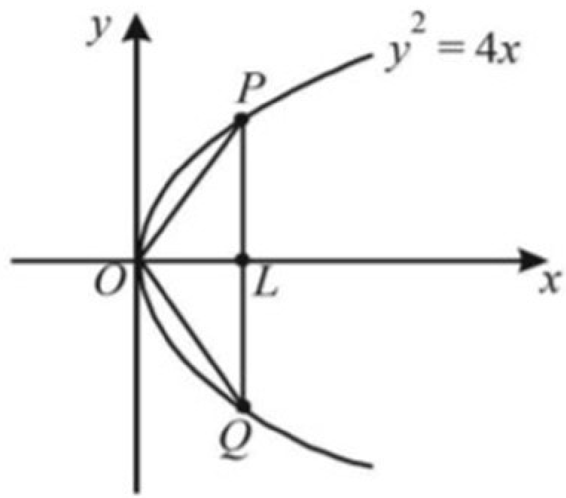
Solution: -

4. Let the equation of chord OP be $y = mx$.

Then eq^n of chord OQ will be $y = -\frac{1}{m}x$ $\quad [\because OQ \perp OP]$

P is pt. of intersection of $y = mx$ and $y^2 = 4x$.

Solving the two we get $P\left(\frac{4}{m^2}, \frac{4}{m}\right)$



Q is pt. of intersection of $y = -\frac{1}{m}x$ and $y^2 = 4x$.

Solving the two we get $Q\,(4m^2, -4m)$

Now eq. of PQ is,

$$y+4m=\frac{\frac{4}{m}+4m}{\frac{4}{m^2}-4m^2}(x-4m^2)$$

$$\Rightarrow \quad y+4m=\frac{m}{1-m^2}(x-4m^2)$$

$$\Rightarrow \quad (1-m^2)\,y+4m-4m^3=mx-4m^3$$

$$\Rightarrow \quad mx-(1-m^2)\,y-4m=0$$

This line meets x-axis where $y=0$ i.e. $x=4$

$\Rightarrow \quad OL=4$, which is constant as independent of m. Again let (h, k) be the mid pt. of PQ, then

$$h=\frac{4m^2+\frac{4}{m^2}}{2} \text{ and } k=\frac{\frac{4}{m}-4m}{2}$$

$$\Rightarrow \quad h=2\left(m^2+\frac{1}{m^2}\right) \text{ and } k=2\left(\frac{1}{m}-m\right)$$

$$\Rightarrow \quad h=2\left[\left(\frac{1}{m}-m\right)+2\right] \Rightarrow \quad h=2\left[\frac{k^2}{4}+2\right]$$

$$\Rightarrow \quad 2h=k^2+8 \quad \Rightarrow \quad k^2=2\,(h-4)$$

$\therefore \quad$ Locus of (h, k) is $y^2=2\,(x-4)$