3. Three normals are drawn from the point (c, 0) to the curve $y^2 = x$. Show that c must be greater than 1/2. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other. ***(1991 - 4 Marks)***

Solution: -

3. Given parabola is $y^2 = x$.

Normal is $y = mx - \frac{m}{2} - \frac{m^3}{4}$

As per question this normal passes through $(c, 0)$ therefore, we get

$$mc - \frac{m}{2} - \frac{m^3}{4} = 0 \qquad ...(1)$$

$$\Rightarrow \quad m\left[c - \frac{1}{2} - \frac{m^2}{4}\right] = 0 \Rightarrow \quad m = 0 \text{ or } m^2 = 4\left(c - \frac{1}{2}\right)$$

$m = 0$ shows normal is $y = 0$ i.e. x-axis is always a normal.

Also $m^2 \geq 0 \Rightarrow 4\left(c - \frac{1}{2}\right) \geq 0 \Rightarrow c \geq 1/2$

At $c = \frac{1}{2}$, from (1) $m = 0$

$\therefore$ for other real values of m, $c > 1/2$

Now for other two normals to be perpendicular to each other, we must have $m_1.m_2 = -1$

Or in other words, if m_1, m_2 are roots of $\frac{m^2}{4} + \frac{1}{2} - c = 0$, then product of roots = – 1

$$\Rightarrow \quad \frac{\left(\frac{1}{2} - c\right)}{1/4} = -1 \Rightarrow \frac{1}{2} - c = -\frac{1}{4} \Rightarrow \quad c = 3/4$$