## 7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

**$P_0$ :** $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

**$P_1$ :** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

**$P_2$ :** $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

**$P_3$ :** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

**$P_4$ :** $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that $P_4$ is a particular case of $P_3$)

**$P_5$ :** $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

**$P_6$ :** $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

**$P_7$ :** (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f$ is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if $f$ is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

**Proof of $P_0$** It follows directly by making the substitution $x = t$.

**Proof of $P_1$** Let F be anti derivative of $f$. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

Here, we observe that, if $a = b$, then $\int_a^a f(x)\,dx = 0$.

**Proof of $P_2$** Let F be anti derivative of $f$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{... (1)}$$

$$\int_a^c f(x)\,dx = F(c) - F(a) \quad \text{... (2)}$$

and

$$\int_c^b f(x)\,dx = F(b) - F(c) \quad \text{... (3)}$$

Adding (2) and (3), we get $\int_a^c f(x)dx + \int_c^b f(x)dx = F(b) - F(a) = \int_a^b f(x)dx$

This proves the property $P_2$.

**Proof of $P_3$** Let $t = a + b - x$. Then $dt = -dx$. When $x = a$, $t = b$ and when $x = b$, $t = a$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \quad \text{(by } P_1\text{)}$$

$$= \int_a^b f(a+b-x)\,dx \text{ by } P_0$$

**Proof of $P_4$** Put $t = a - x$. Then $dt = -dx$. When $x = 0, t = a$ and when $x = a, t = 0$. Now proceed as in $P_3$.

**Proof of $P_5$** Using $P_2$, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$.

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -dx$. When $x = a$, $t = a$ and when $x = 2a$, $t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

**Proof of $P_6$** Using $P_5$, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ... (1)

Now, if $f(2a - x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

and if $f(2a - x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

**Proof of $P_7$** Using $P_2$, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx. \text{ Then}$$

Let $t = -x$ in the first integral on the right hand side. $dt = -dx$. When $x = -a$, $t = a$ and when $x = 0$, $t = 0$. Also $x = -t$.





Therefore $\int_{-a}^a f(x)\,dx = -\int_a^0 f(-t)\,dt + \int_0^a f(x)\,dx$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad \text{(by } P_0\text{)} \quad ... (1)$$

(i) Now, if $f$ is an even function, then $f(-x) = f(x)$ and so (1) becomes

$$\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

(ii) If $f$ is an odd function, then $f(-x) = -f(x)$ and so (1) becomes

$$\int_{-a}^a f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx = 0$$

Therefore $\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(by } P_0\text{) ... (1)}$$

(i) Now, if $f$ is an even function, then $f(-x) = f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If $f$ is an odd function, then $f(-x) = -f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

**Example 30** Evaluate $\int_{-1}^{2} \left| x^3 - x \right| dx$

**Solution** We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by $P_2$ we write

$$\int_{-1}^{2} \left| x^3 - x \right| dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

**Example 31** Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$

**Solution** We observe that $\sin^2 x$ is an even function. Therefore, by $P_7$ (i), we get

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2}\,dx = \int_0^{\frac{\pi}{4}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{\pi}{4} - \frac{1}{2}$$

**Example 32** Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

**Solution** Let $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$. Then, by $P_4$, we have

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$$

or $$2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

or $$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Therefore, (by $P_1$) we get

$$I = \frac{-\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\int_0^{1} \frac{dt}{1+t^2} \text{ (by } P_7\text{, since } \frac{1}{1+t^2} \text{ is even function)}$$

$$= \pi\left[\tan^{-1} t\right]_0^1 = \pi\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

**Example 33** Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$

**Solution** Let $I = \int_{-1}^{1} \sin^5 x \cos^4 x\,dx$. Let $f(x) = \sin^5 x \cos^4 x$. Then

$f(-x) = \sin^5(-x)\cos^4(-x) = -\sin^5 x \cos^4 x = -f(x)$, i.e., $f$ is an odd function. Therefore, by $P_7$ (ii), $I = 0$

**Example 34** Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

**Solution** Let I = $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ... (1)

Then, by $P_4$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 (\frac{\pi}{2} - x)}{\sin^4(\frac{\pi}{2} - x) + \cos^4(\frac{\pi}{2} - x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx \quad ... (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence $\quad I = \frac{\pi}{4}$

**Example 35** Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

**Solution** Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}$ ... (1)

Then, by $P_3$ $\quad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}\, dx}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad ... (2)$$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ Hence } I = \frac{\pi}{12}$$

**Example 36** Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

**Solution** Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then, by $P_4$

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x + \log 2 - \log 2) \, dx \text{ (by adding and subtracting log 2)}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx \quad \text{(Why?)}$$

Put $2x = t$ in the first integral. Then $2\,dx = dt$, when $x = 0, t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

Therefore $$2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2}\log 2$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2}\log 2 \quad [\text{by } P_6 \text{ as } \sin(\pi - t) = \sin t)$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2}\log 2 \quad \text{(by changing variable } t \text{ to } x)$$

$$= I - \frac{\pi}{2}\log 2$$

Hence $$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{-\pi}{2}\log 2.$$

## EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\, dx}{\sin^5 x + \cos^5 x}$

5. $\int_{-5}^{5} |x+2|\, dx$

6. $\int_2^8 |x-5|\, dx$

7. $\int_0^1 x(1-x)^n\, dx$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

9. $\int_0^2 x\sqrt{2-x}\, dx$

10. $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\, dx$

11. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

12. $\int_0^{\pi} \frac{x\, dx}{1+\sin x}$

13. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$

14. $\int_0^{2\pi} \cos^5 x\, dx$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}\, dx$

16. $\int_0^{\pi} \log(1+\cos x)\, dx$

17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$

18. $\int_0^4 |x-1|\, dx$

19. Show that $\int_0^a f(x)g(x)\, dx = 2\int_0^a f(x)\, dx$, if $f$ and $g$ are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 (B) 2 (C) $\pi$ (D) 1

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 (B) $\frac{3}{4}$ (C) 0 (D) –2

## *Miscellaneous Examples*

**Example 37** Find $\int \cos 6x \sqrt{1+\sin 6x}\, dx$

**Solution** Put $t = 1 + \sin 6x$, so that $dt = 6 \cos 6x\, dx$

Therefore $$\int \cos 6x \sqrt{1+\sin 6x}\, dx = \frac{1}{6}\int t^{\frac{1}{2}} dt$$

$$= \frac{1}{6} \times \frac{2}{3} (t)^{\frac{3}{2}} + \text{C} = \frac{1}{9}(1+\sin 6x)^{\frac{3}{2}} + \text{C}$$

**Example 38** Find $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx$

**Solution** We have $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx = \int \frac{(1-\frac{1}{x^3})^{\frac{1}{4}}}{x^4}\, dx$

Put $1-\frac{1}{x^3} = 1 - x^{-3} = t$, so that $\frac{3}{x^4}\, dx = dt$

Therefore $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx = \frac{1}{3}\int t^{\frac{1}{4}}\, dt = \frac{1}{3} \times \frac{4}{5}\, t^{\frac{5}{4}} + \text{C} = \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}} + \text{C}$

**Example 39** Find $\int \frac{x^4\, dx}{(x-1)(x^2+1)}$

**Solution** We have

$$\frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{x^3 - x^2 + x - 1}$$

$$= (x+1) + \frac{1}{(x-1)(x^2+1)} \qquad \text{... (1)}$$

Now express $$\frac{1}{(x-1)(x^2+1)} = \frac{\text{A}}{(x-1)} + \frac{\text{B}x+\text{C}}{(x^2+1)} \qquad \text{... (2)}$$

So $$1 = A(x^2+1) + (Bx + C)(x-1)$$
$$= (A+B)x^2 + (C-B)x + A - C$$

Equating coefficients on both sides, we get A + B = 0, C – B = 0 and A – C = 1, which give $A = \frac{1}{2}, B = C = -\frac{1}{2}$. Substituting values of A, B and C in (2), we get

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \qquad \text{... (3)}$$

Again, substituting (3) in (1), we have

$$\frac{x^4}{(x-1)(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

Therefore

$$\int \frac{x^4}{(x-1)(x^2+x+1)}dx = \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

**Example 40** Find $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

**Solution** Let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

$$= \int \log(\log x)\, dx + \int \frac{1}{(\log x)^2}\, dx$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$I = x\log(\log x) - \int \frac{1}{x\log x} x\, dx + \int \frac{dx}{(\log x)^2}$$

$$= x\log(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \qquad \text{... (1)}$$

Again, consider $\int \frac{dx}{\log x}$, take 1 as the second function and integrate it by parts, we have $\int \frac{dx}{\log x} = \left[\frac{x}{\log x} - \int x\left\{-\frac{1}{(\log x)^2}\left(\frac{1}{x}\right)\right\} dx\right]$ ... (2)

Putting (2) in (1), we get

$$\mathrm{I} = x \log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2} = x \log(\log x) - \frac{x}{\log x} + \mathrm{C}$$

**Example 41** Find $\int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx$

**Solution** We have

$$\mathrm{I} = \int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx = \int \sqrt{\tan x}\,(1 + \cot x)\, dx$$

Put $\tan x = t^2$, so that $\sec^2 x\, dx = 2t\, dt$

or $$dx = \frac{2t\, dt}{1+t^4}$$

Then $$\mathrm{I} = \int t\left(1 + \frac{1}{t^2}\right)\frac{2t}{(1+t^4)}\, dt$$

$$= 2\int \frac{(t^2+1)}{t^4+1}\, dt = 2\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t^2+\frac{1}{t^2}\right)} = 2\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2 + 2}$$

Put $t - \frac{1}{t} = y$, so that $\left(1+\frac{1}{t^2}\right) dt = dy$. Then

$$\mathrm{I} = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2} = \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}} + \mathrm{C} = \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + \mathrm{C}$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + \mathrm{C} = \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \mathrm{C}$$

**Example 42** Find $\int \frac{\sin 2x \cos 2x\, dx}{\sqrt{9 - \cos^4(2x)}}$

**Solution** Let $\mathrm{I} = \int \frac{\sin 2x \cos 2x}{\sqrt{9-\cos^4 2x}}\, dx$

Put $\cos^2(2x) = t$ so that $4 \sin 2x \cos 2x\, dx = -dt$

Therefore $$I = -\frac{1}{4}\int \frac{dt}{\sqrt{9-t^2}} = -\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right) + C = -\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + C$$

**Example 43** Evaluate $\int_{-1}^{\frac{3}{2}} \left| x \sin(\pi x) \right| dx$

**Solution** Here $f(x) = |x \sin \pi x| = \begin{cases} x \sin \pi x \text{ for } -1 \le x \le 1 \\ -x \sin \pi x \text{ for } 1 \le x \le \frac{3}{2} \end{cases}$

Therefore $$\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx = \int_{-1}^{1} x \sin \pi x\, dx + \int_{1}^{\frac{3}{2}} -x \sin \pi x\, dx$$

$$= \int_{-1}^{1} x \sin \pi x\, dx - \int_{1}^{\frac{3}{2}} x \sin \pi x\, dx$$

Integrating both integrals on righthand side, we get

$$\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx = \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$

$$= \frac{2}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3}{\pi} + \frac{1}{\pi^2}$$

**Example 44** Evaluate $\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

**Solution** Let $I = \int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2 \sin^2 x} = \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$ (using $P_4$)

$$= \pi \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} - \int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - I$$

Thus $$2I = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

or $$I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}\ \text{(using } P_6\text{)}$$

$$=\pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}\right]$$

$$=\pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\text{cosec}^2 x\,dx}{a^2\cot^2 x+b^2}\right]$$

$$=\pi\left[\int_0^{1}\frac{dt}{a^2+b^2t^2}-\int_1^{0}\frac{du}{a^2u^2+b^2}\right](put\ \tan x=t\ and\ \cot x=u)$$

$$=\frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1-\frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0=\frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a}+\tan^{-1}\frac{a}{b}\right]=\frac{\pi^2}{2ab}$$

## *Miscellaneous Exercise on Chapter 7*

Integrate the functions in Exercises 1 to 24.

**1.** $\frac{1}{x-x^3}$ **2.** $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$ **3.** $\frac{1}{x\sqrt{ax-x^2}}$ [Hint: Put $x=\frac{a}{t}$]

**4.** $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$ **5.** $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$, put $x=t^6$]

**6.** $\frac{5x}{(x+1)(x^2+9)}$ **7.** $\frac{\sin x}{\sin(x-a)}$ **8.** $\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$

**9.** $\frac{\cos x}{\sqrt{4-\sin^2 x}}$ **10.** $\frac{\sin^8-\cos^8 x}{1-2\sin^2 x\cos^2 x}$ **11.** $\frac{1}{\cos(x+a)\cos(x+b)}$

**12.** $\frac{x^3}{\sqrt{1-x^8}}$ **13.** $\frac{e^x}{(1+e^x)(2+e^x)}$ **14.** $\frac{1}{(x^2+1)(x^2+4)}$

**15.** $\cos^3 x\ e^{\log \sin x}$ **16.** $e^{3\log x}(x^4+1)^{-1}$ **17.** $f'(ax+b)\,[f(ax+b)]^n$

**18.** $\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$ **19.** $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}$, $x\in[0,1]$