Miscellaneous examples

Example # 16 : Find the area contained between the two arms of curves $(y - x)^2 = x^3$ between x = 0 and x = 1.

Solution $(y - x)^2 = x^3 \Rightarrow y = x \pm x^{3/2}$

For arm

$$y = x + x^{3/2} \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{2}x^{1/2} > 0 \qquad x \geq 0.$$

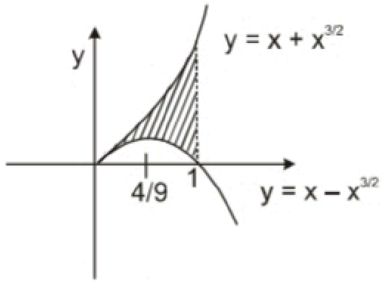
y is increasing function.

For arm

$$y = x - x^{3/2} \Rightarrow \frac{dy}{dx} = 1 - \frac{3}{2}x^{1/2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{4}{9}, \frac{d^2y}{dx^2} = -\frac{3}{4}x^{-\frac{1}{2}} < 0 \text{ at } x = \frac{4}{9}$$

$\therefore$ at $x = \frac{4}{9}$, $y = x - x^{3/2}$ has maxima.



Figure

Required area $= \int_0^1 (x + x^{3/2} - x + x^{3/2})dx$

$$= 2\int_0^1 x^{3/2}dx = \left.\frac{2\,x^{5/2}}{5/2}\right]_0^1 \qquad = \frac{4}{5}$$

0 π/4

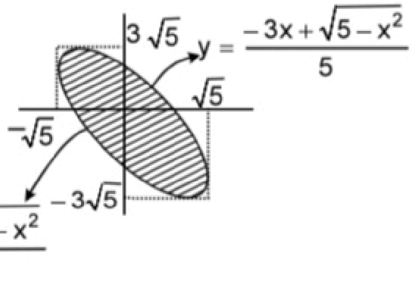
Example # 15 : Find area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$

Solution :

$$5y^2 + 6xy + 2x^2 - 1 = 0$$

$$y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2 - 1)}}{10}$$

$$y = \frac{-3x \pm \sqrt{5 - x^2}}{5}$$



$\because$ y is real $\Rightarrow$ R.H.S. is also real.

$\Rightarrow$ $-\sqrt{5} \le x \le \sqrt{5}$

If $x = -\sqrt{5}$, $y = 3\sqrt{5}$

If $x = \sqrt{5}$, $y = -3\sqrt{5}$

If $x = 0$, $y = \pm \frac{1}{\sqrt{5}}$

If $y = 0$, $x = \pm \frac{1}{\sqrt{2}}$

Required area $= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x + \sqrt{5 - x^2}}{5} - \frac{-3x - \sqrt{5 - x^2}}{5} \right) dx$

$$= \frac{2}{5}\int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{5-x^2}\,dx = \frac{4}{5}\int_{0}^{\sqrt{5}} \sqrt{5-x^2}\,dx$$

Put x $= \sqrt{5}\sin\theta : dx = \sqrt{5}\cos\theta\, d\theta$

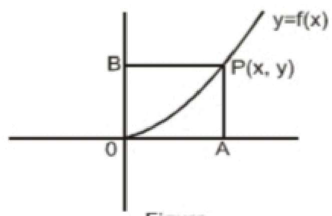
L.L : x $= 0 \Rightarrow \theta = 0$

U.L : x $= \sqrt{5} \Rightarrow \theta = \frac{\pi}{2}$ $= \frac{4}{5}\int_{\theta=0}^{\frac{\pi}{2}} \sqrt{5-5\sin^2\theta}\ \sqrt{5}\cos\theta d\theta = 4\int_{0}^{\frac{\pi}{2}} \cos^2\theta d\theta = 4\ \frac{1}{2}\ \frac{\pi}{2} = \pi$

Example # 18: A curve $y = f(x)$ passes through the origin and lies entirely in the first quadrant. Through any point $P(x, y)$ on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in $m : n$, then show that $f(x) = cx^{m/n}$ or $f(x) = cx^{n/m}$ (c-being arbitrary).

Solution Area (OAPB) = xy

Area (OAPO) = $\int_0^x f(t)dt$



Figure

Area (OPBO) = $xy - \int_0^x f(t)dt$

$$\frac{\text{Area (OAPO)}}{\text{Area (OPBO)}} = \frac{m}{n}$$

$$n\int_0^x f(t)dt = m\left(xy - \int_0^x f(t)dt\right)$$

$$n\int_0^x f(t)dt = mx\,f(x) - m\int_0^x f(t)dt$$

Differentiating w.r.t. x

$nf(x) = m\,f(x) + mx\,f'(x) - m\,f(x)$

$$\frac{f'(x)}{f(x)} = \frac{n}{m}\frac{1}{x}$$

$f(x) = cx^{n/m}$

similarly $f(x) = cx^{m/n}$