7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the area of the light shaded region

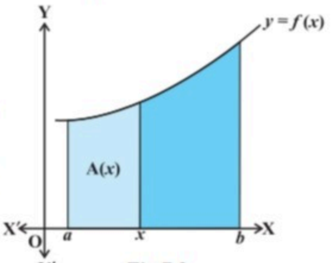

Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by A(x). We call the function A(x) as *Area function* and is given by

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \qquad \text{... (1)}$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function. Then $\mathbf{A'(x) = f(x)}$, for all $\boldsymbol{x \in [a, b]}$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx =$ (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$. For instance, the consideration of definite integral $\int_{-2}^3 x(x^2-1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$\int_a^b f(x)\,dx = [\mathrm{F}(x) + \mathrm{C}]_a^b = [\mathrm{F}(b) + \mathrm{C}] - [\mathrm{F}(a) + \mathrm{C}] = \mathrm{F}(b) - \mathrm{F}(a)$.

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate $\mathrm{F}(b) - \mathrm{F}(a) = [\mathrm{F}(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\,dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$

(iii) $\int_1^2 \frac{x\,dx}{(x+1)(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$

Solution

(i) Let $\mathrm{I} = \int_2^3 x^2\,dx$. Since $\int x^2\,dx = \frac{x^3}{3} = \mathrm{F}(x)$,

Therefore, by the second fundamental theorem, we get

$$\mathrm{I} = \mathrm{F}(3) - \mathrm{F}(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $\mathrm{I} = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

Thus, $\int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right] = \mathrm{F}(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(9) - F(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $I = \int_1^2 \frac{x\,dx}{(x+1)(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $\int \frac{x\,dx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = F(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(2) - F(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\, dt$. Consider $\int \sin^3 2t \cos 2t\, dt$

Put $\sin 2t = u$ so that $2\cos 2t\, dt = du$ or $\cos 2t\, dt = \frac{1}{2}\, du$

So $\int \sin^3 2t \cos 2t\, dt = \frac{1}{2}\int u^3 du$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = F(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$
2. $\int_{2}^{3}\frac{1}{x}\,dx$
3. $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$
4. $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$
5. $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$
6. $\int_{4}^{5}e^x\,dx$
7. $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$
8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\operatorname{cosec} x\,dx$
9. $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$
10. $\int_{0}^{1}\frac{dx}{1+x^2}$
11. $\int_{2}^{3}\frac{dx}{x^2-1}$
12. $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$
13. $\int_{2}^{3}\frac{x\,dx}{x^2+1}$
14. $\int_{0}^{1}\frac{2x+3}{5x^2+1}\,dx$
15. $\int_{0}^{1}x\,e^{x^2}\,dx$
16. $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$
17. $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$
18. $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$
19. $\int_{0}^{2}\frac{6x+3}{x^2+4}\,dx$
20. $\int_{0}^{1}(x\,e^x+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

21. $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

22. $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_{a}^{b}f(x)\,dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y=f(x)$ or $x=g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

Note In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

Example 28 Evaluate $\int_{-1}^{1}5x^4\sqrt{x^5+1}\,dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\,dx$.

Therefore, $\int 5x^4\sqrt{x^5+1}\,dx = \int\sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$

Hence, $\int_{-1}^{1}5x^4\sqrt{x^5+1}\,dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$

Example 28 Evaluate $\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\, dx$.

Therefore, $\int 5x^4 \sqrt{x^5+1}\, dx = \int \sqrt{t}\, dt = \frac{2}{3} t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$

Hence,
$$\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5\,x^4\, dx$.
Note that, when $x = -1, t = 0$ and when $x = 1, t = 2$
Thus, as x varies from -1 to 1, t varies from 0 to 2

Therefore
$$\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx = \int_0^2 \sqrt{t}\, dt$$
$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2})$$

Example 29 Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

Solution Let $t = \tan^{-1} x$, then $dt = \frac{1}{1+x^2}\, dx$. The new limits are, when $x = 0, t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\frac{\pi}{4}$.

Therefore
$$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} t\, dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

Let $t = x^5 + 1$. Then $dt = 5x^4\, dx$.

Note that, when $x = -1,\ t = 0$ and when $x = 1,\ t = 2$

Thus, as x varies from -1 to 1, t varies from 0 to 2

Therefore $$\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx = \int_0^2 \sqrt{t}\, dt$$

$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Example 29 Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

Solution Let $t = \tan^{-1}x$, then $dt = \dfrac{1}{1+x^2}\, dx$. The new limits are, when $x = 0, t = 0$ and when $x = 1,\ t = \dfrac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\dfrac{\pi}{4}$.

Therefore $$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} t\, dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

1. $\displaystyle\int_0^1 \frac{x}{x^2+1}\, dx$ **2.** $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$ **3.** $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

4. $\displaystyle\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$

6. $\displaystyle\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\displaystyle\int_{-1}^1 \frac{dx}{x^2+2x+5}$ **8.** $\displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\displaystyle\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\, dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

10. If $f(x) = \displaystyle\int_0^x t \sin t\, dt$, then $f'(x)$ is

(A) $\cos x + x \sin x$ (B) $x \sin x$

(C) $x \cos x$ (D) $\sin x + x \cos x$