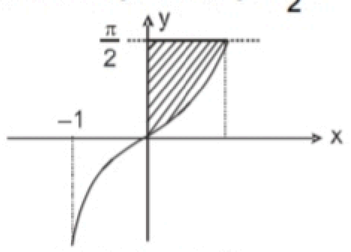
Example # 6 : Find area bounded between $y = \sin^{-1}x$ and y–axis between y = 0 and $y = \frac{\pi}{2}$.

Solution $y = \sin^{-1} x \quad \Rightarrow \quad x = \sin y$



Graph of $y = \sin^{-1}x$

$$\text{Required area} = \int_0^{\frac{\pi}{2}} \sin y \, dy$$

$$= -\cos y \Big]_0^{\frac{\pi}{2}} = -(0 - 1) = 1$$

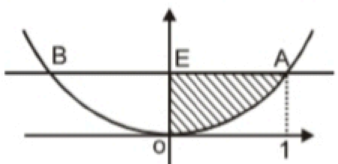
Note : The area in above example can also evaluated by integration with respect to x.

Area = (area of rectangle formed by x = 0, y = 0 , x = 1, $y = \frac{\pi}{2}$) – (area bounded by $y = \sin^{-1}x$, x–axis between x = 0 and x = 1)

$$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - \left(x \sin^{-1} x + \sqrt{1 - x^2} \right)_0^1 = \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1 \right) = 1$$

Example # 7 : Find the area bounded by the parabola $x^2 = y$, y-axis and the line y = 1.

Solution Graph of $y = x^2$



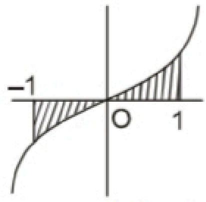
$$\text{Area OEBO} = \text{Area OAEO} = \int_0^1 |x| \, dy = \int_0^1 \sqrt{y} \, dy = \frac{2}{3}$$

Example # 5 : Find the area bounded by $y = x^3$ and x- axis between ordinates $x = -1$ and $x = 1$

Solution Required area = $\int_{-1}^{0} -x^3 dx + \int_{0}^{1} x^3 dx$

$$= \left[-\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{4}\right]_{0}^{1}$$

$$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$$



Graph of $y = x^3$

Note : Most general formula for area bounded by curve $y = f(x)$ and x- axis between ordinates $x = a$ and $x = b$ is

$$\int_{a}^{b} |f(x)| dx$$

Example # 9 : For any real t, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.

Solution It is a point on hyperbola $x^2 - y^2 = 1$.

$$\text{Area (PQRP)} = 2\int_{1}^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx = 2\int_{1}^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2-1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ell n(x+\sqrt{x^2-1})\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}} = \frac{e^{2t_1}-e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta OPQ = 2\times\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) = \frac{e^{2t_1}+e^{-2t_1}}{4} - t_1.$$

$\therefore$ Required area = area ΔOPQ – area (PQRP)

$= t_1$

(b) If $g(y) \le 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $-\int_{y=c}^{d} g(y)dy$

Note : General formula for area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} |g(y)|\,dy$

Example # 14 : Find the area of the region bounded by y = sin x, y = cos x and ordinates x = 0, x = $\pi/2$

Solution $\int_0^{\pi/2} |\sin x - \cos x|\,dx$

$$\int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx = 2(\sqrt{2} - 1)$$

Example # 15 : Find area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$

Solution : $5y^2 + 6xy + 2x^2 - 1 = 0$

$$y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2 - 1)}}{10}$$

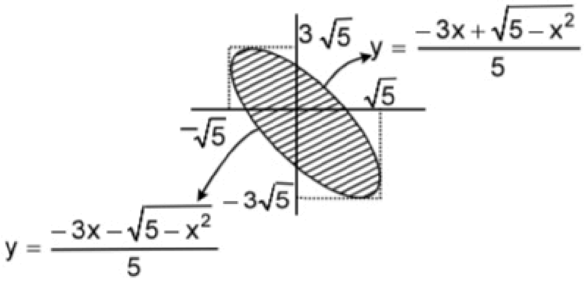
$$y = \frac{-3x \pm \sqrt{5 - x^2}}{5}$$

$\because$ y is real $\Rightarrow$ R.H.S. is also real.

$\Rightarrow$ $-\sqrt{5} \le x \le \sqrt{5}$

If $x = -\sqrt{5}$, $y = 3\sqrt{5}$

If $x = \sqrt{5}$, $y = -3\sqrt{5}$



Example # 13 : Find the area enclosed by curve (graph) $y = x^2 + x + 1$ and its tangent at (1,3) between ordinates $x = -1$ and $x = 1$.

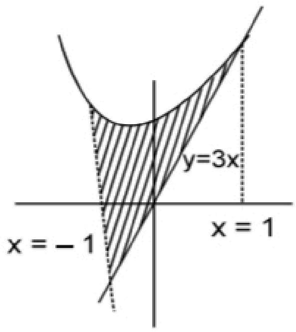
Solution $\frac{dy}{dx} = 2x + 1$

$\frac{dy}{dx} = 3$ at $x = 1$

Equation of tangent is

$y - 3 = 3(x - 1)$

$y = 3x$



Required area $= \int_{-1}^{1} (x^2 + x + 1 - 3x)\, dx$

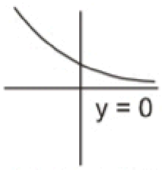
$= \int_{-1}^{1} (x^2 - 2x + 1)\, dx = \left.\frac{x^3}{3} - x^2 + x\right]_{-1}^{1}$

$= \left(\frac{1}{3} - 1 + 1\right) - \left(-\frac{1}{3} - 1 - 1\right) = \frac{2}{3} + 2 = \frac{8}{3}$

Note : Area bounded by curves $y = f(x)$ and $y = g(x)$ between ordinates $x = a$ and $x = b$ is $\int_a^b |f(x) - g(x)|\, dx$.

Example # 10 : Find asymptote of $y = e^{-x}$

Solution $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{-x} = 0$

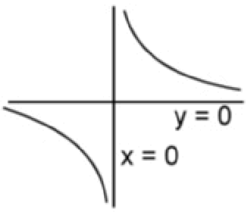


Graph of $y = e^{-x}$

$\therefore$ y = 0 is asymptote.

Example # 11 : Find asymptotes of xy = 1 and draw graph.

Solution : $y = \dfrac{1}{x}$



$\lim_{x\to 0} y = \lim_{x\to 0} \dfrac{1}{x} = \infty \Rightarrow x = 0$ is asymptote.

$\lim_{x\to \infty} y = \lim_{x\to \infty} \dfrac{1}{x} = 0 \Rightarrow y = 0$ is asymptote.

0 -0 -

Example # 17 : Let A (m) be area bounded by parabola $y = x^2 + 2x - 3$ and the line $y = mx + 1$. Find the least area A(m).

Solution Solving we obtain

$x^2 + (2 - m)x - 4 = 0$

Let α, β be roots $\Rightarrow \alpha + \beta = m - 2$, $\alpha\beta = -4$

$$A(m) = \left|\int_\alpha^\beta (mx+1-x^2-2x+3)dx\right| = \left|\int_\alpha^\beta (-x^2+(m-2)x+4)dx\right|$$

$$= \left|\left(-\frac{x^3}{3}+(m-2)\frac{x^2}{2}+4x\right)_\alpha^\beta\right| = \left|\frac{\alpha^3-\beta^3}{3}+\frac{m-2}{2}(\beta^2-\alpha^2)+4(\beta-\alpha)\right|$$

$$= |\beta-\alpha| . \left|-\frac{1}{3}(\beta^2+\beta\alpha+\alpha^2)+\frac{(m-2)}{2}(\beta+\alpha)+4\right|$$

$$= \sqrt{(m-2)^2+16}\left|-\frac{1}{3}\left((m-2)^2+4\right)+\frac{(m-2)}{2}(m-2)+4\right| = \sqrt{(m-2)^2+16}\left|\frac{1}{6}(m-2)^2+\frac{8}{3}\right|$$

$$A(m) = \frac{1}{6}\left((m-2)^2 + 16\right)^{3/2}$$

$$\text{Least } A(m) = \frac{1}{6}(16)^{3/2} = \frac{32}{3}.$$