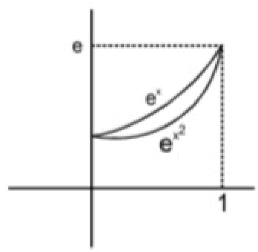
Example # 17 Estimate the value of $\int_0^1 e^{x^2} dx$ by using $\int_0^1 e^x dx$.



Solution For $x \in (0, 1)$, $e^{x^2} < e^x$

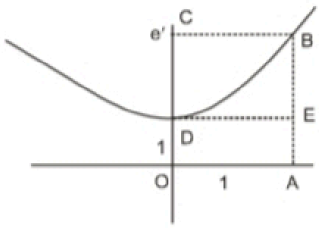
$\Rightarrow \quad 1 \times 1 < \int_0^1 e^{x^2} dx < \int_0^1 e^x dx$

$1 < \int_0^1 e^{x^2} dx < e - 1$

Example # 16 Estimate the value of $\int_0^1 e^{x^2} dx$ using (i) rectangle, (ii) triangle.

Solution (i) By using rectangle

Area OAED < $\int_0^1 e^{x^2} dx$ < Area OABC



$1 < \int_0^1 e^{x^2} dx < 1 . e$

$1 < \int_0^1 e^{x^2} dx < e$

(ii) By using triangle

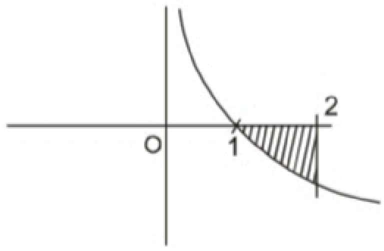
Area OAED < $\int_0^1 e^{x^2} dx$ < Area OAED + Area of triangle DEB

$1 < \int_0^1 e^{x^2} dx < 1 + \frac{1}{2} . 1. (e - 1)$ $\qquad 1 < \int_0^1 e^{x^2} dx < \frac{e+1}{2}$

Graph of y = f(x)

Example # 4 : Find area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between x = 1 and x = 2

Solution: A rough graph of $y = \log_{\frac{1}{2}} x$ is as follows



$$\text{Area} = -\int_1^2 \log_{\frac{1}{2}} x \; dx = -\int_1^2 \log_e x \cdot \log_{\frac{1}{2}} e \; dx$$

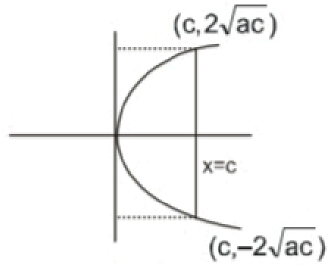
$$= -\log_{\frac{1}{2}} e \cdot [x\log_e x - x]_1^2$$

$$= -\log_{\frac{1}{2}} e \cdot (2\log_e 2 - 2 - 0 + 1) = -\log_{\frac{1}{2}} e \cdot (2\log_e 2 - 1)$$

Note : If y = f(x) does not change sign in [a, b], then area bounded by y = f(x), x-axis between ordinates x = a, x = b is $\left| \int_a^b f(x)\,dx \right|$

Example # 3 The area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by the double ordinate and its distance from the vertex. Find the value of k ?

Solution Consider $y^2 = 4ax$, $a > 0$ and $x = c$



Figure

Area by double ordinate = $2\int_0^c 2\sqrt{a}\sqrt{x}\,dx = \frac{8}{3}\sqrt{a}\,c^{3/2}$

Area by double ordinate = k (Area of rectangle)

$\frac{8}{3}\sqrt{a}\,c^{3/2} = k\,4\sqrt{a}\,c^{3/2}$

$k = \frac{2}{3}$

Example # 2 : Find area bounded by the curve $y = \ell n\, x + \tan^{-1} x$ and x-axis between ordinates x = 1 and x = 2.

Solution $y = \ell n\, x + \tan^{-1}x$

Domain $x > 0,\ \frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$

y is increasing and x = 1, $y = \frac{\pi}{4} \Rightarrow$ y is positive in [1, 2]

$\therefore$ Required area = $\int_1^2 (\ell n\, x + \tan^{-1} x)\, dx$

$$= \left[x\, \ell n\, x - x + x \tan^{-1} x - \frac{1}{2} \ell n\, (1+x^2) \right]_1^2$$

$$= 2\, \ell n\, 2 - 2 + 2 \tan^{-1}2 - \frac{1}{2}\, \ell n\, 5 - 0 + 1 - \tan^{-1} 1 + \frac{1}{2}\, \ell n\, 2$$

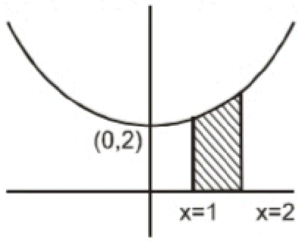
$$= \frac{5}{2}\, \ell n\, 2 - \frac{1}{2}\, \ell n\, 5 + 2 \tan^{-1}2 - \frac{\pi}{4} - 1$$

Note : If a function is known to be positive valued then graph is not necessary.

Example # 1 : Find the area enclosed between the curve $y = x^2 + 2$, x-axis, $x = 1$ and $x = 2$.

Solution

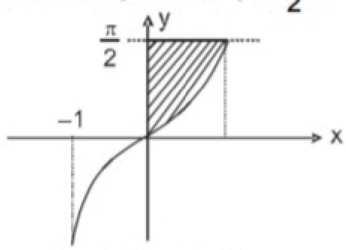
Graph of $y = x^2 + 2$



$$\text{Area} = \int_1^2 (x^2 + 2)\,dx = \left[\frac{x^3}{3} + 2x\right]_1^2 = \frac{13}{3}$$

Example # 6 : Find area bounded between $y = \sin^{-1}x$ and y–axis between $y = 0$ and $y = \frac{\pi}{2}$.

Solution $y = \sin^{-1} x \quad \Rightarrow \quad x = \sin y$



Graph of $y = \sin^{-1}x$

Required area $= \int_0^{\frac{\pi}{2}} \sin y \, dy$

$= -\cos y \Big]_0^{\frac{\pi}{2}} = -(0-1) = 1$

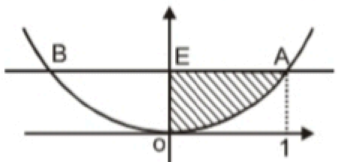
Note : The area in above example can also evaluated by integration with respect to x.

Area = (area of rectangle formed by $x = 0$, $y = 0$, $x = 1$, $y = \frac{\pi}{2}$) – (area bounded by $y = \sin^{-1}x$, x–axis between $x = 0$ and $x = 1$)

$$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - \left(x\sin^{-1} x + \sqrt{1-x^2} \right)_0^1 = \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1 \right) = 1$$

Example # 7 : Find the area bounded by the parabola $x^2 = y$, y-axis and the line $y = 1$.

Solution Graph of $y = x^2$



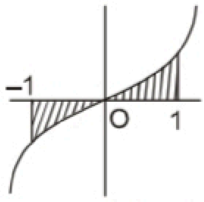
Area OEBO = Area OAEO = $\int^1 |x| \, dy = \int^1 \sqrt{y} \, dy = \frac{2}{3}$

Example # 5 : Find the area bounded by $y = x^3$ and x- axis between ordinates $x = -1$ and $x = 1$

Solution Required area = $\int_{-1}^{0} -x^3 dx + \int_{0}^{1} x^3 dx$

$$= \left[-\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{4}\right]_{0}^{1}$$

$$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$$



Graph of $y = x^3$

Note : Most general formula for area bounded by curve $y = f(x)$ and x- axis between ordinates $x = a$ and $x = b$ is

$$\int_{a}^{b} |f(x)|\, dx$$

Example # 9 : For any real t, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.

Solution It is a point on hyperbola $x^2 - y^2 = 1$.

$$\text{Area (PQRP)} = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2-1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ell n(x+\sqrt{x^2-1})\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}} = \frac{e^{2t_1}-e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta OPQ = 2\times\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) = \frac{e^{2t_1}+e^{-2t_1}}{4} - t_1.$$

$\therefore$ Required area = area ΔOPQ – area (PQRP)
$= t_1$

(b) If $g(y) \le 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $-\int_{y=c}^{d} g(y)dy$

Note : General formula for area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} |g(y)|\,dy$