Proposition 4.1 (Pigeonhole's Principle I) *If we distribute n pigeons into $n - 1$ cages (the* pigeonholes*), then at least one cage will necessarily contain at least two pigeons.*

[1]After the German mathematician of the nineteenth century Gustav L. Dirichlet. Actually, we shall have to wait until Sect. 7.2 (cf. Lemma 7.7) to appreciate the original problem that motivated Dirichlet to introduce the pigeonhole principle.

Proof By contraposition, if each one of the $n-1$ cages was to contain at most one pigeon, then we would have at most $n-1$ pigeons. □

As naive as it seems, the pigeonhole principle allows us to establish astonishing consequences, as shown by the coming examples. In all that follows, we shall always assume that the relation of getting acquainted to someone else is *symmetric*, i.e., if A knows B, then B also knows A.

Example 4.2 There are n guests in a party. Show that we can always find two of them who, within the party, are acquainted with the same number of people.

Proof Firstly, note that each of the n guests is acquainted with at least 0 and at most $n-1$ other ones within the party. Therefore, there are two cases to consider:

(i) Each guest knows at least one other person in the party: take $n-1$ rooms, numbered from 1 to $n-1$, and put in room i all of the guests (if any) who know exactly i other guests. Since we have $n-1$ rooms (the pigeonholes) and n guests (the pigeons), Dirichlet's principle assures that at least one room will contain at least two people. By the way we have allocated the guests in the rooms, these two guests know, within the party, the same number of other people.

(ii) There exists at least one "guest" who, actually, has no acquaintances within the party: then, no guest knows all of the other $n-1$ people, so that, in this case, we can number the rooms from 0 to $n-2$ and reason as in (i). Once more, pigeonhole's principle guarantees that at least one of the rooms will contain at least two people. Also as in (i), these two people know, within the party, the same number of other people.

□

Example 4.3 (IMO Shortlist) Each subset of the set $\{1, 2, \ldots, 10\}$ is painted with one, our of n possible colors. Find the greatest possible value of n for which one can always find two distinct and nonempty sets $A, B \subset \{1, 2, \ldots, 10\}$, such that A, B and $A \cup B$ are all painted with the same color.

Solution Let $X = \{1, 2, \ldots, 10\}$. Given ten colors $C_1, C_2, \ldots, C_{10}$, let us paint the k-element subsets of X with color C_k, for $1 \leq k \leq 10$. This way, given two distinct and nonempty subsets A and B of X, there are two possibilities: either A and B have different numbers of elements, and hence were painted with different colors, or A and B have the same number of elements; in this second case, $A \cup B$ has more elements than A and B, and thus was painted with a color different from that of A and B. Therefore, we conclude that $n \leq 9$.

Now, take only nine colors, and let $A_i = \{1, \ldots, i\}$ for $1 \leq i \leq 10$. Since

$$A_1 \subset A_2 \subset \cdots \subset A_{10}$$

is a *chain* of ten distinct and nonempty subsets of X (the pigeons) but we have only nine colors (the pigeonholes), Dirichlet's principle assures the existence of indices $1 \leq i < j \leq 10$ such that A_i and A_j are painted with a single color. Since $A_i \cup A_j = A_j$, it suffices to let $A_i = A$ and $A_j = B$ to fulfill the stated conditions. Thus, the largest possible value of n is 9. □

There are several interesting generalizations of the pigeonhole principle, and the coming result collects the one who is probably the most important of them. For what is to come, recall (cf. the paragraph preceding Problem 7, page 41) that the **integer part** of a real number x, denoted $\lfloor x \rfloor$, is the greatest integer which is less than or equal to x. Thus, if $\lfloor x \rfloor = n \in \mathbb{Z}$, then $n \leq x < n + 1$. It is frequently useful to write down such inequalities as

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1. \tag{4.1}$$

Proposition 4.4 (Pigeonhole's Principle II) *If n pigeons are distributed among k different cages, then at least one cage will contain at least $\left\lfloor \frac{n-1}{k} \right\rfloor + 1$ pigeons.*

Proof Arguing once more by contraposition, let us suppose that each of the k cages contains at most $\left\lfloor \frac{n-1}{k} \right\rfloor$ pigeons. Then, altogether we have at most $k \left\lfloor \frac{n-1}{k} \right\rfloor$ pigeons, and it suffices to note that, by the left inequality in (4.1),

$$k \left\lfloor \frac{n-1}{k} \right\rfloor \leq k \left(\frac{n-1}{k} \right) = n - 1 < n.$$

□

Let us examine a few other examples.

Example 4.5 Prove that, in any group of twenty people, at least three of them were born in the same day of the week.

Proof Let the twenty people be the pigeons and the 7 days of the week be the cages. Then, associate a person to a day of the week if he/she was born in that day. The second version of the pigeonhole principle guarantees that at least one cage will contain at least $\left\lfloor \frac{20-1}{7} \right\rfloor + 1 = 3$ people. Thus, these three people were born in the same day of the week. □

Example 4.6 (Leningrad[2]) Each 1×1 square of a 5×41 chessboard is painted either red or blue. Prove that it is possible to choose three lines and three columns of the chessboard so that the nine 1×1 squares into which they intersect are painted with the same color.

Proof Since we have used only two colors, the second version of the pigeonhole principle implies that, in each column (of five 1×1 squares), one of the two colors must occur at least $\lfloor \frac{5-1}{2} \rfloor + 1 = 3$ times. Call such a color *dominant* and, for each

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[2]Former name of the Russian city of Saint Petersburg.

of the 41 columns, take note of its dominant color. Since one out of the two colors dominates in each of the 41 columns, by invoking once more the second version of the pigeonhole principle we conclude that one of the two colors is the dominant one in at least $\lfloor \frac{41-1}{2} \rfloor + 1 = 21$ columns. Assume, without loss of generality, that such a color is red, and call a column as *red* if red is its dominant color. Then, choose 21 red columns and, in each of them, choose three 1×1 red squares.

Notice that there are exactly $\binom{5}{2} = 10$ possible ways of choosing three of the five 1×1 squares of each column. Therefore, out of the 21 red columns (and once again from the second version of the pigeonhole principle), the three 1×1 chosen red squares occupy exactly the same positions in at least $\left\lfloor \frac{21-1}{10} \right\rfloor + 1 = 3$ columns.

In other words, we have at least three columns having 1×1 red squares in the same set of three lines, and this is exactly what we were asked to prove. □