EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2 (2x + 5)$
2. $\sin 3x \cos 4x$
3. $\cos 2x \cos 4x \cos 6x$
4. $\sin^3 (2x + 1)$
5. $\sin^3 x \cos^3 x$
6. $\sin x \sin 2x \sin 3x$
7. $\sin 4x \sin 8x$
8. $\dfrac{1-\cos x}{1+\cos x}$
9. $\dfrac{\cos x}{1+\cos x}$
10. $\sin^4 x$
11. $\cos^4 2x$
12. $\dfrac{\sin^2 x}{1+\cos x}$
13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
14. $\dfrac{\cos x - \sin x}{1+\sin 2x}$
15. $\tan^3 2x \sec 2x$
16. $\tan^4 x$
17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
19. $\dfrac{1}{\sin x \cos^3 x}$
20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$
21. $\sin^{-1} (\cos x)$
22. $\dfrac{1}{\cos (x-a) \cos (x-b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

    (A) $\tan x + \cot x + C$ (B) $\tan x + \text{cosec } x + C$

    (C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

24. $\displaystyle\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

    (A) $-\cot (ex^x) + C$ (B) $\tan (xe^x) + C$

    (C) $\tan (e^x) + C$ (D) $\cot (e^x) + C$

7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$

(2) $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

(3) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

(4) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

(5) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

(6) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

We now prove the above results:

(1) We have $\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a} \left[\frac{(x+a)-(x-a)}{(x-a)(x+a)} \right] = \frac{1}{2a} \left[\frac{1}{x-a} - \frac{1}{x+a} \right]$$

Therefore, $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right]$$

$$= \frac{1}{2a} \left[\log |(x-a)| - \log |(x+a)| \right] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a} \left[\frac{(a+x)+(a-x)}{(a+x)(a-x)} \right] = \frac{1}{2a} \left[\frac{1}{a-x} + \frac{1}{a+x} \right]$$

Therefore, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$

$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + C$

$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

Note The technique used in (1) will be explained in Section 7.5.

(3) Put $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

Therefore, $\int \frac{dx}{x^2 + a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$

$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(4) Let $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + C_1$

$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$

$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a| + C_1$

$= \log\left|x + \sqrt{x^2 - a^2}\right| + C$, where $C = C_1 - \log|a|$

(5) Let $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$

$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C$

(6) Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$

$= \int \sec\theta\, d\theta = \log|(\sec\theta + \tan\theta)| + C_1$

$$= \log \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1} \right| + C_1$$

$$= \log \left| x + \sqrt{x^2 + a^2} \right| - \log |a| + C_1$$

$$= \log \left| x + \sqrt{x^2 + a^2} \right| + C \text{, where } C = C_1 - \log |a|$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

(7) To find the integral $\int \frac{dx}{ax^2 + bx + c}$, we write

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Now, put $x + \frac{b}{2a} = t$ so that $dx = dt$ and writing $\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$. We find the integral reduced to the form $\frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$ depending upon the sign of $\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type $\int \frac{px + q}{ax^2 + bx + c}\, dx$, where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A\,(2ax + b) + B$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, we proceed as in (9) and transform the integral into known standard forms.

Let us illustrate the above methods by some examples.

Example 8 Find the following integrals:

(i) $\int \frac{dx}{x^2-16}$ (ii) $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

(i) We have $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$ [by 7.4 (1)]

(ii) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

Put $x - 1 = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C$ [by 7.4 (5)]

$$= \sin^{-1}(x-1) + C$$

Example 9 Find the following integrals :

(i) $\int \frac{dx}{x^2-6x+13}$ (ii) $\int \frac{dx}{3x^2+13x-10}$ (iii) $\int \frac{dx}{\sqrt{5x^2-2x}}$

Solution

(i) We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}\,dx$

Let $x - 3 = t$. Then $dx = dt$

Therefore, $\int \frac{dx}{x^2-6x+13} = \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$ [by 7.4 (3)]

$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

(ii) The given integral is of the form 7.4 (7). We write the denominator of the integrand,

$$3x^2 + 13x - 10 = 3\left(x^2 + \frac{13x}{3} - \frac{10}{3}\right)$$

$$= 3\left[\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right] \text{ (completing the square)}$$

Thus $\displaystyle\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}$

Put $x + \frac{13}{6} = t$. Then $dx = dt$.

Therefore, $\displaystyle\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dt}{t^2 - \left(\frac{17}{6}\right)^2}$

$$= \frac{1}{3 \times 2 \times \frac{17}{6}} \log\left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right| + C_1 \quad \text{[by 7.4 (i)]}$$

$$= \frac{1}{17} \log\left|\frac{x + \frac{13}{6} - \frac{17}{6}}{x + \frac{13}{6} + \frac{17}{6}}\right| + C_1$$

$$= \frac{1}{17} \log\left|\frac{6x - 4}{6x + 30}\right| + C_1$$

$$= \frac{1}{17} \log\left|\frac{3x - 2}{x + 5}\right| + C_1 + \frac{1}{17}\log\frac{1}{3}$$

$$= \frac{1}{17} \log\left|\frac{3x - 2}{x + 5}\right| + C, \text{ where } C = C_1 + \frac{1}{17}\log\frac{1}{3}$$

(iii) We have $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \int \frac{dx}{\sqrt{5\left(x^2 - \frac{2x}{5}\right)}}$

$$= \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} \quad \text{(completing the square)}$$

Put $x - \frac{1}{5} = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}}$

$$= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C \qquad \text{[by 7.4 (4)]}$$

$$= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + C$$

Example 10 Find the following integrals:

(i) $\int \frac{x+2}{2x^2 + 6x + 5} dx$ (ii) $\int \frac{x+3}{\sqrt{5 - 4x + x^2}} dx$

Solution

(i) Using the formula 7.4 (9), we express

$$x + 2 = \text{A} \frac{d}{dx}\left(2x^2 + 6x + 5\right) + \text{B} = \text{A}\,(4x + 6) + \text{B}$$

Equating the coefficients of x and the constant terms from both sides, we get

$4\text{A} = 1$ and $6\text{A} + \text{B} = 2$ or $\text{A} = \frac{1}{4}$ and $\text{B} = \frac{1}{2}$.

Therefore, $\int \frac{x+2}{2x^2 + 6x + 5} = \frac{1}{4} \int \frac{4x+6}{2x^2 + 6x + 5} dx + \frac{1}{2} \int \frac{dx}{2x^2 + 6x + 5}$

$$= \frac{1}{4} \text{I}_1 + \frac{1}{2} \text{I}_2 \quad \text{(say)} \qquad \text{... (1)}$$

In I_1, put $2x^2 + 6x + 5 = t$, so that $(4x + 6)\, dx = dt$

Therefore, $$I_1 = \int \frac{dt}{t} = \log|t| + C_1$$

$$= \log|2x^2 + 6x + 5| + C_1 \qquad \text{... (2)}$$

and $$I_2 = \int \frac{dx}{2x^2 + 6x + 5} = \frac{1}{2}\int \frac{dx}{x^2 + 3x + \frac{5}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Put $x + \frac{3}{2} = t$, so that $dx = dt$, we get

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2 \times \frac{1}{2}} \tan^{-1} 2t + C_2 \qquad \text{[by 7.4 (3)]}$$

$$= \tan^{-1} 2\left(x + \frac{3}{2}\right) + C_2 = \tan^{-1}(2x + 3) + C_2 \qquad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$\int \frac{x + 2}{2x^2 + 6x + 5}\, dx = \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}(2x + 3) + C$$

where, $$C = \frac{C_1}{4} + \frac{C_2}{2}$$

(ii) This integral is of the form given in 7.4 (10). Let us express

$$x + 3 = A\frac{d}{dx}(5 - 4x - x^2) + B = A(-4 - 2x) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$-2A = 1$ and $-4A + B = 3$, i.e., $A = -\frac{1}{2}$ and $B = 1$