Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\int \mathrm{P}\,dx}$.

(iii) Write the solution of the given differential equation as

$$y\ (\mathrm{I.F}) = \int (\mathrm{Q} \times \mathrm{I.F})\,dx + \mathrm{C}$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{\mathrm{P}_1 dy}$ and the solution of the differential equation is given by

$$x\ .\ (\mathrm{I.F}) = \int (\mathrm{Q}_1 \times \mathrm{I.F})\,dy + \mathrm{C}$$

Example 19 Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$.

Solution Given differential equation is of the form

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}, \text{ where } \mathrm{P} = -1 \text{ and } \mathrm{Q} = \cos x$$

Therefore $\qquad \mathrm{I.F} = e^{\int -1\, dx} = e^{-x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

or

$$\frac{dy}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$$

On integrating both sides with respect to x, we get

$$ye^{-x} = \int e^{-x}\cos x\,dx + \mathrm{C} \qquad \text{... (1)}$$

Let

$$\mathrm{I} = \int e^{-x}\cos x\,dx$$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int (-\sin x)\,(-e^{-x})\,dx$$

$$= -\cos x\, e^{-x} - \int \sin x\, e^{-x}\, dx$$

$$= -\cos x\, e^{-x} - \left[\sin x(-e^{-x}) - \int \cos x\,(-e^{-x})\, dx\right]$$

$$= -\cos x\, e^{-x} + \sin x\, e^{-x} - \int \cos x\, e^{-x}\, dx$$

or $$\text{I} = -e^{-x} \cos x + \sin x\, e^{-x} - \text{I}$$

or $$2\text{I} = (\sin x - \cos x)\, e^{-x}$$

or $$\text{I} = \frac{(\sin x - \cos x)\, e^{-x}}{2}$$

Substituting the value of I in equation (1), we get

$$y e^{-x} = \left(\frac{\sin x - \cos x}{2}\right) e^{-x} + \text{C}$$

or $$y = \left(\frac{\sin x - \cos x}{2}\right) + \text{C}\, e^{x}$$

which is the general solution of the given differential equation.

Example 20 Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2 \; (x \neq 0)$.

Solution The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2 \qquad \ldots (1)$$

Dividing both sides of equation (1) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type $\frac{dy}{dx} + \text{P}y = \text{Q}$, where $\text{P} = \frac{2}{x}$ and $\text{Q} = x$.

So $$\text{I.F} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2 \; [as\; e^{\log f(x)} = f(x)]$$

Therefore, solution of the given equation is given by

$$y \,.\, x^2 = \int (x)(x^2)\, dx + \text{C} = \int x^3 dx + \text{C}$$

or $$y = \frac{x^2}{4} + \text{C}\, x^{-2}$$

which is the general solution of the given differential equation.

Example 21 Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type $\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$, where $\mathrm{P}_1 = -\frac{1}{y}$ and $\mathrm{Q}_1 = 2y$. Therefore $\mathrm{I.F} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log (y)^{-1}} = \frac{1}{y}$

Hence, the solution of the given differential equation is

$$x\frac{1}{y} = \int (2y)\left(\frac{1}{y}\right) dy + \mathrm{C}$$

or $$\frac{x}{y} = \int (2dy) + \mathrm{C}$$

or $$\frac{x}{y} = 2y + \mathrm{C}$$

or $$x = 2y^2 + \mathrm{C}y$$

which is a general solution of the given differential equation.

Example 22 Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x + x^2 \cot x \ (x \neq 0)$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

Solution The given equation is a linear differential equation of the type $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$, where $\mathrm{P} = \cot x$ and $\mathrm{Q} = 2x + x^2 \cot x$. Therefore

$$\mathrm{I.F} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Hence, the solution of the differential equation is given by

$$y \,.\, \sin x = \int (2x + x^2 \cot x) \sin x \, dx + \mathrm{C}$$

or $$y \sin x = \int 2x \sin x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = \sin x \left(\frac{2x^2}{2}\right) - \int \cos x \left(\frac{2x^2}{2}\right) dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x - \int x^2 \cos x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x + C \qquad \ldots (1)$$

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in equation (1), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

or $$C = \frac{-\pi^2}{4}$$

Substituting the value of C in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or $$y = x^2 - \frac{\pi^2}{4 \sin x} \ (\sin x \neq 0)$$

which is the particular solution of the given differential equation.

Example 23 Find the equation of a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

Solution We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x \qquad \ldots (1)$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

Therefore, $$\text{I.F} = e^{\int -x \, dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x) \left(e^{\frac{-x^2}{2}}\right) dx + \text{C} \quad \text{... (2)}$$

Let $\qquad \text{I} = \int (x)\, e^{\frac{-x^2}{2}}\, dx$

Let $\frac{-x^2}{2} = t$, then $-\, x\, dx = dt$ or $x\, dx = -\, dt$.

Therefore, $\qquad \text{I} = -\int e^t\, dt = -e^t = -\, e^{\frac{-x^2}{2}}$

Substituting the value of I in equation (2), we get

$$y\, e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + \text{C}$$

or $$y = -1 + \text{C}\, e^{\frac{x^2}{2}} \quad \text{... (3)}$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through (0, 1). Substituting $x = 0$ and $y = 1$ in equation (3) we get

$$1 = -\, 1 + \text{C}\,.\, e^0 \quad \text{or} \quad \text{C} = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2\, e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

EXERCISE 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1. $\frac{dy}{dx} + 2y = \sin x$ **2.** $\frac{dy}{dx} + 3y = e^{-2x}$ **3.** $\frac{dy}{dx} + \frac{y}{x} = x^2$

4. $\frac{dy}{dx} + (\sec x)\, y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$ **5.** $\cos^2 x \frac{dy}{dx} + y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$

6. $x\frac{dy}{dx} + 2y = x^2 \log x$ **7.** $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$

8. $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx\ (x \ne 0)$