17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q} \qquad \text{... (1)}$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x)\frac{dy}{dx} + \mathrm{P}.(g(x))\, y = \mathrm{Q}\,.\, g(x) \qquad \text{... (2)}$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y \cdot g(x)$.

i.e. $$g(x)\frac{dy}{dx} + \text{P}.\,g(x)\,y = \frac{d}{dx}[y \cdot g(x)]$$

or $$g(x)\frac{dy}{dx} + \text{P}.\,g(x)\,y = g(x)\frac{dy}{dx} + y\,g'(x)$$

$\Rightarrow$ $$\text{P}.\,g(x) = g'(x)$$

or $$\text{P} = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int \text{P}dx = \int \frac{g'(x)}{g(x)}\,dx$$

or $$\int \text{P}\cdot dx = \log(g(x))$$

or $$g(x) = e^{\int \text{P}\,dx}$$

On multiplying the equation (1) by $g(x) = e^{\int \text{P}\,dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int \text{P}\,dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of $g(x)$ in equation (2), we get

$$e^{\int \text{P}\,dx}\frac{dy}{dx} + \text{P}e^{\int \text{P}\,dx}y = \text{Q}\cdot e^{\int \text{P}\,dx}$$

or $$\frac{d}{dx}\left(ye^{\int \text{P}dx}\right) = \text{Q}e^{\int \text{P}dx}$$

Integrating both sides with respect to x, we get

$$y \cdot e^{\int \text{P}dx} = \int\left(\text{Q}.e^{\int \text{P}\,dx}\right)dx$$

or $$y = e^{-\int \text{P}\,dx}.\int\left(\text{Q}.e^{\int \text{P}\,dx}\right)dx + \text{C}$$

which is the general solution of the differential equation.

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + \text{P}y = \text{Q}$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\int \text{P}\,dx}$.

(iii) Write the solution of the given differential equation as

$$y\ (\text{I.F}) = \int (\text{Q} \times \text{I.F})\,dx + \text{C}$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + \text{P}_1 x = \text{Q}_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{\text{P}_1 dy}$ and the solution of the differential equation is given by

$$x\ .\ (\text{I.F}) = \int (\text{Q}_1 \times \text{I.F})\,dy + \text{C}$$

Example 19 Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$.

Solution Given differential equation is of the form

$$\frac{dy}{dx} + \text{P}y = \text{Q}, \text{ where P} = -1 \text{ and Q} = \cos x$$

Therefore $\qquad \text{I.F} = e^{\int -1\,dx} = e^{-x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

or $$\frac{dy}{dx}\left(y e^{-x}\right) = e^{-x}\cos x$$

On integrating both sides with respect to x, we get

$$ye^{-x} = \int e^{-x}\cos x\,dx + \text{C} \qquad \ldots (1)$$

Let $$\text{I} = \int e^{-x}\cos x\,dx$$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int (-\sin x)\,(-e^{-x})\,dx$$