or $$I = \frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \text{ (using P}_6\text{)}$$

$$= \pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right]$$

$$= \pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\text{cosec}^2 x\,dx}{a^2\cot^2 x + b^2}\right]$$

$$= \pi\left[\int_0^1\frac{dt}{a^2 + b^2t^2} - \int_1^0\frac{du}{a^2u^2 + b^2}\right]\left(put\tan x = t\, and \cot x = u\right)$$

$$= \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1 - \frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0 = \frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a} + \tan^{-1}\frac{a}{b}\right] = \frac{\pi^2}{2ab}$$

Miscellaneous Exercise on Chapter 7

Integrate the functions in Exercises 1 to 24.

1. $\frac{1}{x - x^3}$ **2.** $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$ **3.** $\frac{1}{x\sqrt{ax - x^2}}$ [Hint: Put $x = \frac{a}{t}$]

4. $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$ **5.** $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{6}}\right)}$, put $x = t^6$]

6. $\frac{5x}{(x+1)(x^2+9)}$ **7.** $\frac{\sin x}{\sin(x - a)}$ **8.** $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

9. $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$ **10.** $\frac{\sin^8 - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$ **11.** $\frac{1}{\cos(x+a)\cos(x+b)}$

12. $\frac{x^3}{\sqrt{1 - x^8}}$ **13.** $\frac{e^x}{(1+e^x)(2+e^x)}$ **14.** $\frac{1}{(x^2+1)(x^2+4)}$

15. $\cos^3 x\, e^{\log \sin x}$ **16.** $e^{3\log x}(x^4 + 1)^{-1}$ **17.** $f'(ax + b)[f(ax + b)]^n$

18. $\frac{1}{\sqrt{\sin^3 x\sin(x + \alpha)}}$ **19.** $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$, $x \in [0, 1]$

Put $x + 1 = y$ so that $dx = dy$.

Thus $\int \sqrt{3-2x-x^2}\, dx = \int \sqrt{4-y^2}\, dy$

$$= \frac{1}{2} y \sqrt{4-y^2} + \frac{4}{2} \sin^{-1} \frac{y}{2} + C \qquad \text{[using 7.6.2 (iii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2 \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

1. $\sqrt{4-x^2}$ **2.** $\sqrt{1-4x^2}$ **3.** $\sqrt{x^2+4x+6}$

4. $\sqrt{x^2+4x+1}$ **5.** $\sqrt{1-4x-x^2}$ **6.** $\sqrt{x^2+4x-5}$

7. $\sqrt{1+3x-x^2}$ **8.** $\sqrt{x^2+3x}$ **9.** $\sqrt{1+\frac{x^2}{9}}$

Choose the correct answer in Exercises 10 to 11.

10. $\int \sqrt{1+x^2}\, dx$ is equal to

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$ (C) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2 \log\left|x+\sqrt{1+x^2}\right| + C$

11. $\int \sqrt{x^2-8x+7}\, dx$ is equal to

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

Thus, $$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Example 22 Find (i) $\int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx$ (ii) $\int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx$

Solution

(i) We have $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx$

Consider $f(x) = \tan^{-1} x$, then $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx = e^x \tan^{-1} x + C$

(ii) We have $I = \int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx = \int e^x [\frac{x^2 - 1 + 1 + 1)}{(x+1)^2}]\, dx$

$$= \int e^x [\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]\, dx = \int e^x [\frac{x-1}{x+1} + \frac{2}{(x+1)^2}]\, dx$$

Consider $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $$\int \frac{x^2+1}{(x+1)^2} e^x\, dx = \frac{x-1}{x+1} e^x + C$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$ **2.** $x \sin 3x$ **3.** $x^2 e^x$ **4.** $x \log x$

5. $x \log 2x$ **6.** $x^2 \log x$ **7.** $x \sin^{-1} x$ **8.** $x \tan^{-1} x$

9. $x \cos^{-1} x$ **10.** $(\sin^{-1} x)^2$ **11.** $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$ **12.** $x \sec^2 x$

13. $\tan^{-1} x$ **14.** $x (\log x)^2$ **15.** $(x^2 + 1) \log x$

16. $e^x(\sin x + \cos x)$ **17.** $\frac{x\,e^x}{(1+x)^2}$ **18.** $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

19. $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$ **20.** $\frac{(x-3)\,e^x}{(x-1)^3}$ **21.** $e^{2x}\sin x$

22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\,dx$ equals

(A) $\frac{1}{3}e^{x^3}+C$ (B) $\frac{1}{3}e^{x^2}+C$

(C) $\frac{1}{2}e^{x^3}+C$ (D) $\frac{1}{2}e^{x^2}+C$

24. $\int e^x \sec x\,(1+\tan x)\,dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

7.6.2 *Integrals of some more types*

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int\sqrt{x^2-a^2}\,dx$ (ii) $\int\sqrt{x^2+a^2}\,dx$ (iii) $\int\sqrt{a^2-x^2}\,dx$

(i) Let $I=\int\sqrt{x^2-a^2}\,dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$I = x\sqrt{x^2-a^2} - \int\frac{1}{2}\frac{2x}{\sqrt{x^2-a^2}}\,x\,dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$