Therefore, $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence, $$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx = (\sin^{-1} x)\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\,(-\sqrt{1-x^2})\,dx$$

$$= -\sqrt{1-x^2}\,\sin^{-1} x + x + C = x - \sqrt{1-x^2}\,\sin^{-1} x + C$$

Alternatively, this integral can also be worked out by making substitution $\sin^{-1} x = \theta$ and then integrating by parts.

Example 21 Find $\int e^x \sin x\,dx$

Solution Take e^x as the first function and $\sin x$ as second function. Then, integrating by parts, we have

$$I = \int e^x \sin x\,dx = e^x(-\cos x) + \int e^x \cos x\,dx$$

$$= -e^x \cos x + I_1 \text{ (say)} \qquad \text{... (1)}$$

Taking e^x and $\cos x$ as the first and second functions, respectively, in I_1, we get

$$I_1 = e^x \sin x - \int e^x \sin x\,dx$$

Substituting the value of I_1 in (1), we get

$$I = -e^x \cos x + e^x \sin x - I \quad\text{or}\quad 2I = e^x(\sin x - \cos x)$$

Hence, $$I = \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Alternatively, above integral can also be determined by taking $\sin x$ as the first function and e^x the second function.

7.6.1 *Integral of the type* $\int e^x [f(x) + f'(x)]\,dx$

We have $$I = \int e^x [f(x) + f'(x)]\,dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$

$$= I_1 + \int e^x f'(x)\,dx, \text{ where } I_1 = \int e^x f(x)\,dx \qquad \text{... (1)}$$

Taking $f(x)$ and e^x as the first function and second function, respectively, in I_1 and integrating it by parts, we have $I_1 = f(x)\,e^x - \int f'(x)\,e^x\,dx + C$

Substituting I_1 in (1), we get

$$I = e^x f(x) - \int f'(x)\,e^x\,dx + \int e^x f'(x)\,dx + C = e^x f(x) + C$$

Thus, $$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Example 22 Find (i) $\int e^x (\tan^{-1}x + \frac{1}{1+x^2})\, dx$ (ii) $\int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx$

Solution

(i) We have $I = \int e^x (\tan^{-1}x + \frac{1}{1+x^2})\, dx$

Consider $f(x) = \tan^{-1}x$, then $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $I = \int e^x (\tan^{-1}x + \frac{1}{1+x^2})\, dx = e^x \tan^{-1}x + C$

(ii) We have $I = \int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx = \int e^x [\frac{x^2-1+1+1)}{(x+1)^2}]\, dx$

$$= \int e^x [\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]\, dx = \int e^x [\frac{x-1}{x+1} + \frac{2}{(x+1)^2}]\, dx$$

Consider $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $$\int \frac{x^2+1}{(x+1)^2} e^x\, dx = \frac{x-1}{x+1} e^x + C$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$ **2.** $x \sin 3x$ **3.** $x^2 e^x$ **4.** $x \log x$

5. $x \log 2x$ **6.** $x^2 \log x$ **7.** $x \sin^{-1} x$ **8.** $x \tan^{-1} x$

9. $x \cos^{-1} x$ **10.** $(\sin^{-1}x)^2$ **11.** $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$ **12.** $x \sec^2 x$

13. $\tan^{-1}x$ **14.** $x (\log x)^2$ **15.** $(x^2 + 1) \log x$

16. $e^x (\sin x + \cos x)$ **17.** $\dfrac{x\,e^x}{(1+x)^2}$ **18.** $e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)$

19. $e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$ **20.** $\dfrac{(x-3)\,e^x}{(x-1)^3}$ **21.** $e^{2x}\sin x$

22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3} dx$ equals

(A) $\dfrac{1}{3}e^{x^3} + \text{C}$ (B) $\dfrac{1}{3}e^{x^2} + \text{C}$

(C) $\dfrac{1}{2}e^{x^3} + \text{C}$ (D) $\dfrac{1}{2}e^{x^2} + \text{C}$

24. $\int e^x \sec x\,(1+\tan x)\,dx$ equals

(A) $e^x \cos x + \text{C}$ (B) $e^x \sec x + \text{C}$

(C) $e^x \sin x + \text{C}$ (D) $e^x \tan x + \text{C}$

7.6.2 Integrals of some more types

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int \sqrt{x^2-a^2}\,dx$ (ii) $\int \sqrt{x^2+a^2}\,dx$ (iii) $\int \sqrt{a^2-x^2}\,dx$

(i) Let $\text{I} = \int \sqrt{x^2-a^2}\,dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$\text{I} = x\sqrt{x^2-a^2} - \int \frac{1}{2}\frac{2x}{\sqrt{x^2-a^2}}\,x\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - \text{I} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$2\text{I} = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$\mathbf{I = \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left| x+\sqrt{x^2-a^2}\right| + C}$$

Similarly, integrating other two integrals by parts, taking constant function 1 as the second function, we get

(ii) $$\mathbf{\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left| x+\sqrt{x^2+a^2}\right| + C}$$

(iii) $$\mathbf{\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C}$$

Alternatively, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution $x = a\sec\theta$ in (i), $x = a\tan\theta$ in (ii) and $x = a\sin\theta$ in (iii) respectively.

Example 23 Find $\int \sqrt{x^2+2x+5}\,dx$

Solution Note that

$$\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{(x+1)^2+4}\,dx$$

Put $x + 1 = y$, so that $dx = dy$. Then

$$\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{y^2+2^2}\,dy$$

$$= \frac{1}{2}y\sqrt{y^2+4} + \frac{4}{2}\log\left| y+\sqrt{y^2+4}\right| + C \qquad \text{[using 7.6.2 (ii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left| x+1+\sqrt{x^2+2x+5}\right| + C$$

Example 24 Find $\int \sqrt{3-2x-x^2}\,dx$

Solution Note that $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx$