Question2

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Question 2 Area of a acute ΔABC is Δ and area of it's pedal triangle is p, where $\cos B = \frac{2p}{\Delta}$ and $\sin B = \frac{2\sqrt{3}}{\Delta} \cdot p$. Find the value of $(\cos^2 A + \cos^2 B + \cos^2 C)$.

(a) 1

(b) $\frac{7}{8}$

(c) $\frac{1}{4}$

(d) $\frac{3}{4}$

Solution:

We know that $\tan B = \frac{\sin B}{\cos B}$

$$= \frac{\frac{2\sqrt{3}}{\Delta} \cdot p}{\frac{2p}{\Delta}} = \sqrt{3}$$

$$\Rightarrow \tan B = \tan 60^\circ$$

$$\Rightarrow \boxed{\angle B = 60^\circ} \quad \text{—— (1)}$$

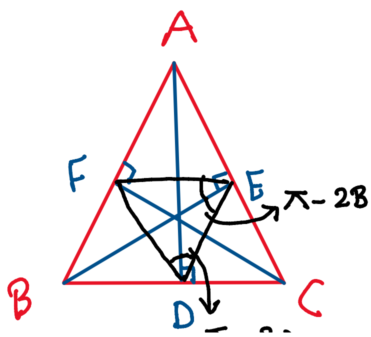
Now, $\cos^2 B + \sin^2 B = 1$

$$\Rightarrow \frac{4p^2}{\Delta^2} + \frac{12p^2}{\Delta^2} = 1$$

$$\Rightarrow \Delta^2 = 16p^2$$

$$\Rightarrow \boxed{\Delta = 4p} \quad \text{—— (2)}$$

Now, $\angle DEF = \pi - 2B = \pi - 2 \times 60 = 60^\circ$



$\therefore$ Using the formula of area of triangle: $\rightarrow$

Area $= \Delta = \frac{1}{2} ab \sin C$

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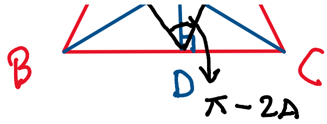

we can write

Area of Pedal triangle

$$P = \frac{1}{2} \times EF \times DE \times \sin(\angle DEF)$$

$$= \frac{1}{2} \times a\cos A \times c\cos C \times \sin 60^\circ$$

$$= \frac{\sqrt{3}}{4} ac \cos A \cdot \cos C \quad \text{--- (3)}$$

Now, Area of triangle $ABC = \Delta = \frac{1}{2} ac \sin B$

$$\Rightarrow ac = \frac{2\Delta}{\sin B} = \frac{2\Delta}{\sin 60^\circ}$$

$$\Rightarrow ac = \frac{4\Delta}{\sqrt{3}} \quad \text{--- (4)}$$

$\therefore$ From equation (3), we can write as: $\rightarrow$

$$P = \frac{\sqrt{3}}{4} \times \frac{4\Delta}{\sqrt{3}} \times \cos A \cos C$$

$$\Rightarrow \cos A \cos C = \frac{P}{\Delta}$$

$$\Rightarrow \cos A \cos C = \frac{1}{4} \quad \text{--- (5)}$$

$\left\{ \begin{array}{l} \text{from equation 2} \\ \Delta = 4p \end{array} \right.$

Now, $A + B + C = \pi$

$$\Rightarrow C = \pi - A - B = \pi - A - 60^\circ$$

$$\Rightarrow C = 120^\circ - A \quad \text{--- (6)}$$

$\therefore$ From equation (5), we can write as: $\rightarrow$

$$\cos A \cdot \cos(120^\circ - A) = \frac{1}{4}$$

$$\Rightarrow \cos A \cdot \left[\cos A \cdot \cos(120^\circ) + \sin A \cdot \sin 120^\circ\right] = \frac{1}{4}$$

$$\Rightarrow \cos A \cdot \left[-\frac{\cos A}{2} + \frac{\sqrt{3}}{2} \sin A\right] = \frac{1}{4}$$

$$\Rightarrow -\cos^2 A + \sqrt{3} \sin A \cos A = \frac{1}{2}$$

$$\Rightarrow -2\cos^2 A + \sqrt{3}(2 \sin A \cos A) = 1$$

$$\Rightarrow (1 - 2\cos^2 A) + \sqrt{3} \sin 2A = 1 + 1$$

$$\Rightarrow \sqrt{3} \sin 2A - \cos 2A = 2$$

$$\Rightarrow \frac{\sqrt{3}}{2} \cdot \sin 2A - \frac{1}{2} \cdot \cos 2A = 1$$

$$\Rightarrow \cos 2A \cdot \cos\frac{\pi}{3} - \sin 2A \cdot \sin\frac{\pi}{3} = -1$$

$$\Rightarrow \cos\left(2A + \frac{\pi}{3}\right) = -1$$

$$\Rightarrow 2A + \frac{\pi}{3} = \pi$$

$$\Rightarrow \boxed{A = \frac{\pi}{3}} \quad \text{—— (7)}$$

$\therefore$ From eq^n (6) :→ $C = 120^\circ - \frac{\pi}{3}$

$$\Rightarrow \boxed{C = \frac{\pi}{3}}$$

$$\therefore \cos^2 A + \cos^2 B + \cos^2 C = \cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{3}$$

$$= \frac{1}{4} \quad \text{Ans}$$

$\boxed{\text{Option (d)}}$