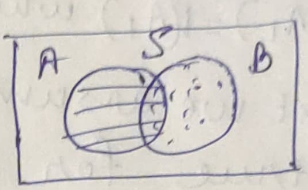
Addition Rule of Probability

Let A and B be any two events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof :



$A \cup B = A \cup (B - (A \cap B))$

So, $P(A \cup B)$

$= P(A) + P(B - (A \cap B))$

$= P(A) + P(B) - P(A \cap B)$

Extension to 3 events :

A, B, C

$P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$= P(A) + P(B) - P(A \cap B) + P(C)$
$\quad - P((A \cap C) \cup (B \cap C))$

$= P(A) + P(B) + P(C) - P(A \cap B)$
$\quad - [P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$
$\quad - P(A \cap C) + P(A \cap B \cap C)$

General Addition Rule : Let $A_1, A_2, \ldots, A_n$ be any events. Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}\sum P(A_i \cap A_j) + \sum_i\sum_j\sum_k P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right) \quad \text{--- (1)}$$

Proof: We will prove the relation ① using the principle of mathematical induction.

For $n=1$, the statement ① becomes $P(A_1)=P(A_1)$ which always holds true.

Next we assume the statement ① to be true for $n=k$. Then we prove it for $n=k+1$.

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)$$

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$

(by addition rule for two events)

$$\cancel{+(-1)^{k+1} P\left(\bigcap_{i=1}^{k} A_i\right) + P(A_{k+1})}$$

$$\cancel{-P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)}$$

$$= \sum_{i=1}^{k} \{P(A_i) - \sum_{i<j}^{k}\sum^{k} P(A_i \cap A_j) + \sum_{i<j<m}^{k}\sum^{k}\sum^{k} P(A_i \cap A_j \cap A_m) - \dots + (-1)^{k+1} P\left(\bigcap_{i=1}^{k} A_i\right) + P(A_{k+1})$$

$$- P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)\Big)$$

$$= \sum_{i=1}^{k+1} P(A_i) - \sum_{i<j}^{k}\sum^{k} (A_i \cap A_j) + \sum_{i<j<m}^{k}\sum^{k}\sum^{k} P(A_i \cap A_j \cap A_m) - \dots + (-1)^{k+1} P\left(\bigcap_{i=1}^{k} A_i\right) - \Big[\sum_{i=1}^{k} P(A_i \cap A_{k+1})$$

$$- \sum_{i<j}^{k}\sum^{k} P((A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1})) + \dots$$

$$+(-1)^{k+1} P\left(\bigcap_{i=1}^{k} (A_i \cap A_{k+1})\right)\Big)$$

$$= \sum_{i=1}^{k+1} P(A_i) - \sum_{i<j}^{k+1}\sum^{k+1} P(A_i \cap A_j) + \sum_{i<j<m}^{k+1}\sum^{k+1}\sum^{k+1} P(A_i \cap A_j \cap A_m) - \dots + (-1)^{k+2} P\left(\bigcap_{i=1}^{k+2} A_i\right)$$

This shows that statement ① is true for $n = k+1$.

Hence by the Principle of Mathematical Induction the General addition rule holds for all n (n is a positive integer).