Example 7.10 At an airport, a person is made to walk through the doorway of a metal detector, for security reasons. If she/he is carrying anything made of metal, the metal detector emits a sound. On what principle does this detector work?

Solution The metal detector works on the principle of resonance in ac circuits. When you walk through a metal detector, you are, in fact, walking through a coil of many turns. The coil is connected to a capacitor tuned so that the circuit is in resonance. When you walk through with metal in your pocket, the impedance of the circuit changes – resulting in significant change in current in the circuit. This change in current is detected and the electronic circuitry causes a sound to be emitted as an alarm.

EXAMPLE 7.10

7.8 LC Oscillations

We know that a capacitor and an inductor can store electrical and magnetic energy, respectively. When a capacitor (initially charged) is connected to an inductor, the charge on the capacitor and the current in the circuit exhibit the phenomenon of electrical oscillations similar to oscillations in mechanical systems (Chapter 14, Class XI).

Let a capacitor be charged q_m (at $t = 0$) and connected to an inductor as shown in Fig. 7.18.

The moment the circuit is completed, the charge on the capacitor starts decreasing, giving rise to current in the circuit. Let q and i be the charge and current in the circuit at time t. Since $\mathrm{d}i/\mathrm{d}t$ is positive, the induced emf in L will have polarity as shown, i.e., $v_b < v_a$. According to Kirchhoff's loop rule,

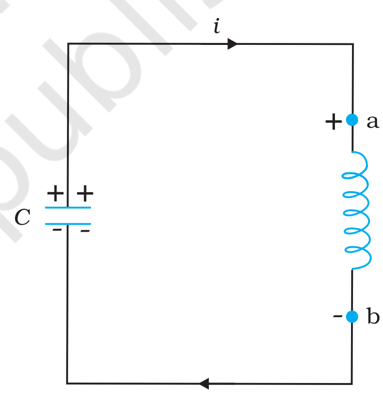

FIGURE 7.18 At the instant shown, the current is increasing; so the polarity of induced emf in the inductor is as shown.

$$\frac{q}{C} - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0 \qquad (7.39)$$

$i = -(\mathrm{d}q/\mathrm{d}t)$ in the present case (as q decreases, i increases). Therefore, Eq. (7.39) becomes:

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{LC}q = 0 \qquad (7.40)$$

This equation has the form $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2\, x = 0$ for a simple harmonic oscillator. The charge on the capacitor, therefore, oscillates with a natural frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad (7.41)$$

and varies sinusoidally with time as

$$q = q_m \cos(\omega_0\, t + \phi) \qquad (7.42)$$

where q_m is the maximum value of q and ϕ is a phase constant. Since $q = q_m$ at $t = 0$, we have $\cos\phi = 1$ or $\phi = 0$. Therefore, in the present case,

$$q = q_m \cos(\omega_0 t) \tag{7.43}$$

The current $i\left(= -\frac{\mathrm{d}q}{\mathrm{d}t}\right)$ is given by

$$i = i_m \sin(\omega_0 t) \tag{7.44}$$

where $i_m = \omega_0 q_m$

Let us now try to visualise how this oscillation takes place in the circuit.

Figure 7.19(a) shows a capacitor with initial charge q_m connected to an ideal inductor. The electrical energy stored in the charged capacitor is $U_E = \frac{1}{2}\frac{q_m^2}{C}$. Since, there is no current in the circuit, energy in the inductor is zero. Thus, the total energy of *LC* circuit is,

$$U = U_E = \frac{1}{2}\frac{q_m^2}{C}$$

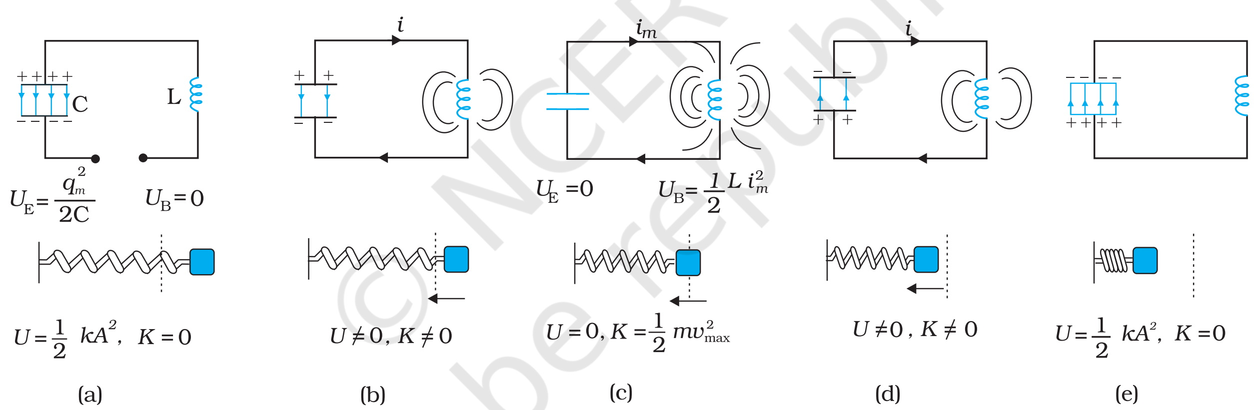

FIGURE 7.19 The oscillations in an *LC* circuit are analogous to the oscillation of a block at the end of a spring. The figure depicts one-half of a cycle.

At $t = 0$, the switch is closed and the capacitor starts to discharge [Fig. 7.19(b)]. As the current increases, it sets up a magnetic field in the inductor and thereby, some energy gets stored in the inductor in the form of magnetic energy: $U_B = (1/2)\, Li^2$. As the current reaches its maximum value i_m, (at $t = T/4$) as in Fig. 7.19(c), all the energy is stored in the magnetic field: $U_B = (1/2)\, Li_m^2$. You can easily check that the maximum electrical energy equals the maximum magnetic energy. The capacitor now has no charge and hence no energy. The current now starts charging the capacitor, as in Fig. 7.19(d). This process continues till the capacitor is fully charged (at $t = T/2$) [Fig. 7.19(e)]. But it is charged with a polarity opposite to its initial state in Fig. 7.19(a). The whole process just described will now repeat itself till the system reverts to its original state. Thus, the energy in the system oscillates between the capacitor and the inductor.

The LC oscillation is similar to the mechanical oscillation of a block attached to a spring. The lower part of each figure in Fig. 7.19 depicts the corresponding stage of a mechanical system (a block attached to a spring). As noted earlier, for a block of a mass m oscillating with frequency ω_0, the equation is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

Here, $\omega_0 = \sqrt{k/m}$, and k is the spring constant. So, x corresponds to q. In case of a mechanical system $F = ma = m\,(dv/dt) = m\,(d^2x/dt^2)$. For an electrical system, $\varepsilon = -L\,(di/dt) = -L\,(d^2q/dt^2)$. Comparing these two equations, we see that L is analogous to mass m: L is a measure of resistance to change in current. In case of LC circuit, $\omega_0 = 1/\sqrt{LC}$ and for mass on a spring, $\omega_0 = \sqrt{k/m}$. So, $1/C$ is analogous to k. The constant k ($=F/x$) tells us the (external) force required to produce a unit displacement whereas $1/C$ ($=V/q$) tells us the potential difference required to store a unit charge. Table 7.1 gives the analogy between mechanical and electrical quantities.

TABLE 7.1 ANALOGIES BETWEEN MECHANICAL AND ELECTRICAL QUANTITIES

Mechanical system	Electrical system
Mass m	Inductance L
Force constant k	Reciprocal capacitance $1/C$
Displacement x	Charge q
Velocity $v = dx/dt$	Current $i = dq/dt$
Mechanical energy	Electromagnetic energy
$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$	$U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$

Note that the above discussion of LC oscillations is not realistic for two reasons:

(i) Every inductor has some resistance. The effect of this resistance is to introduce a damping effect on the charge and current in the circuit and the oscillations finally die away.

(ii) Even if the resistance were zero, the total energy of the system would not remain constant. It is radiated away from the system in the form of electromagnetic waves (discussed in the next chapter). In fact, radio and TV transmitters depend on this radiation.

Two different phenomena, same mathematical treatment

You may like to compare the treatment of a forced damped oscillator discussed in Section 14.10 of Class XI physics textbook, with that of an *LCR* circuit when an ac voltage is applied in it. We have already remarked that Eq. [14.37(b)] of Class XI Textbook is exactly similar to Eq. (7.28) here, although they use different symbols and parameters. Let us therefore list the equivalence between different quantities in the two situations:

Forced oscillations	*Driven LCR circuit*
$m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = F\cos\omega_d\ t$	$L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = v_m \sin\omega t$
Displacement, x	Charge on capacitor, q
Time, t	Time, t
Mass, m	Self inductance, L
Damping constant, b	Resistance, R
Spring constant, k	Inverse capacitance, $1/C$
Driving frequency, ω_d	Driving frequency, ω
Natural frequency of oscillations, ω	Natural frequency of LCR circuit, ω_0
Amplitude of forced oscillations, A	Maximum charge stored, q_m
Amplitude of driving force, F_0	Amplitude of applied voltage, v_m

You must note that since x corresponds to q, the amplitude A (maximum displacement) will correspond to the maximum charge stored, q_m. Equation [14.39 (a)] of Class XI gives the amplitude of oscillations in terms of other parameters, which we reproduce here for convenience:

$$A = \frac{F_0}{\left\{m^2(\omega^2 - \omega_d^2)^2 + \omega_d^2 b^2\right\}^{1/2}}$$

Replace each parameter in the above equation by the corresponding electrical quantity, and see what happens. Eliminate L, C, ω, and ω_0, using $X_L = \omega L$, $X_C = 1/\omega C$, and $\omega_0^2 = 1/LC$. When you use Eqs. (7.33) and (7.34), you will see that there is a perfect match.

You will come across numerous such situations in physics where diverse physical phenomena are represented by the same mathematical equation. If you have dealt with one of them, and you come across another situation, you may simply replace the corresponding quantities and interpret the result in the new context. We suggest that you may try to find more such parallel situations from different areas of physics. One must, of course, be aware of the differences too.

Example 7.11 Show that in the free oscillations of an *LC* circuit, the sum of energies stored in the capacitor and the inductor is constant in time.

Solution Let q_0 be the initial charge on a capacitor. Let the charged capacitor be connected to an inductor of inductance *L*. As you have studied in Section 7.8, this LC circuit will sustain an oscillation with frquency

$$\omega \left(= 2\pi\nu = \frac{1}{\sqrt{LC}} \right)$$

At an instant *t*, charge *q* on the capacitor and the current *i* are given by:

$q(t) = q_0 \cos \omega t$

$i(t) = -q_0\, \omega \sin \omega t$

Energy stored in the capacitor at time *t* is

$$U_E = \frac{1}{2} C\, V^2 = \frac{1}{2}\frac{q^2}{C} = \frac{q_0^2}{2C} \cos^2(\omega t)$$

Energy stored in the inductor at time *t* is

$$U_M = \frac{1}{2} L\, i^2$$

$$= \frac{1}{2} L\, q_0^2\, \omega^2 \sin^2(\omega t)$$

$$= \frac{q_0^2}{2C} \sin^2(\omega t) \qquad \left(\because \omega = 1/\sqrt{LC}\right)$$

Sum of energies

$$U_E + U_M = \frac{q_0^2}{2C}\left(\cos^2 \omega t + \sin^2 \omega t\right)$$

$$= \frac{q_0^2}{2C}$$

This sum is constant in time as q_o and *C*, both are time-independent. Note that it is equal to the initial energy of the capacitor. Why it is so? Think!

Example 7.11

7.9 Transformers

For many purposes, it is necessary to change (or transform) an alternating voltage from one to another of greater or smaller value. This is done with a device called *transformer* using the principle of mutual induction.

A transformer consists of two sets of coils, insulated from each other. They are wound on a soft-iron core, either one on top of the other as in Fig. 7.20(a) or on separate limbs of the core as in Fig. 7.20(b). One of the coils called the *primary coil* has N_p turns. The other coil is called the *secondary coil*; it has N_s turns. Often the primary coil is the input coil and the secondary coil is the output coil of the transformer.