7.6.1 Phasor-diagram solution

From the circuit shown in Fig. 7.12, we see that the resistor, inductor and capacitor are in series. Therefore, the ac current in each element is the same at any time, having the same amplitude and phase. Let it be

$$i = i_m \sin(\omega t+\phi) \tag{7.21}$$

where ϕ is the phase difference between the voltage across the source and the current in the circuit. On the basis of what we have learnt in the previous sections, we shall construct a phasor diagram for the present case.

Let **I** be the phasor representing the current in the circuit as given by Eq. (7.21). Further, let $\mathbf{V_L}$, $\mathbf{V_R}$, $\mathbf{V_C}$, and **V** represent the voltage across the inductor, resistor, capacitor and the source, respectively. From previous section, we know that $\mathbf{V_R}$ is parallel to **I**, $\mathbf{V_C}$ is $\pi/2$ behind **I** and $\mathbf{V_L}$ is $\pi/2$ ahead of **I**. $\mathbf{V_L}$, $\mathbf{V_R}$, $\mathbf{V_C}$ and **I** are shown in Fig. 7.13(a) with apppropriate phase-relations.

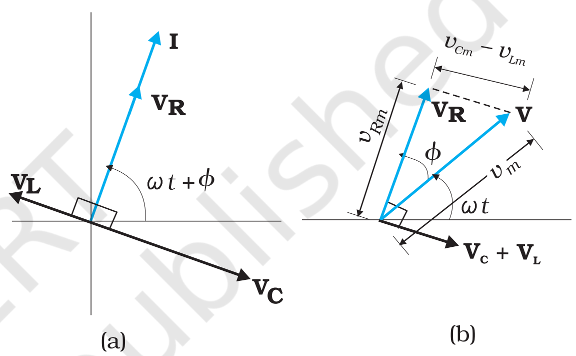

FIGURE 7.13 (a) Relation between the phasors $\mathbf{V}_L$, $\mathbf{V}_R$, $\mathbf{V}_C$, and I, (b) Relation between the phasors $\mathbf{V}_L$, $\mathbf{V}_R$, and $(\mathbf{V}_L + \mathbf{V}_C)$ for the circuit in Fig. 7.12.

The length of these phasors or the amplitude of $\mathbf{V_R}$, $\mathbf{V_C}$ and $\mathbf{V_L}$ are:

$$v_{Rm} = i_m R,\ v_{Cm} = i_m X_C,\ v_{Lm} = i_m X_L \tag{7.22}$$

The voltage Equation (7.20) for the circuit can be written as

$$v_L + v_R + v_C = v \tag{7.23}$$

The phasor relation whose vertical component gives the above equation is

$$\mathbf{V_L} + \mathbf{V_R} + \mathbf{V_C} = \mathbf{V} \tag{7.24}$$

This relation is represented in Fig. 7.13(b). Since $\mathbf{V_C}$ and $\mathbf{V_L}$ are always along the same line and in opposite directions, they can be combined into a single phasor $(\mathbf{V_C} + \mathbf{V_L})$ which has a magnitude $|v_{Cm} - v_{Lm}|$. Since **V** is represented as the hypotenuse of a right-triangle whose sides are $\mathbf{V_R}$ and $(\mathbf{V_C} + \mathbf{V_L})$, the pythagorean theorem gives:

$$v_m^2 = v_{Rm}^2 + \left(v_{Cm} - v_{Lm}\right)^2$$

Substituting the values of v_{Rm}, v_{Cm}, and v_{Lm} from Eq. (7.22) into the above equation, we have

$$v_m^2 = (i_m R)^2 + (i_m X_C - i_m X_L)^2$$

$$= i_m^2 \left[R^2 + (X_C - X_L)^2\right]$$

or, $$i_m = \frac{v_m}{\sqrt{R^2 + (X_C - X_L)^2}} \tag{7.25(a)}$$

By analogy to the resistance in a circuit, we introduce the *impedance Z* in an ac circuit:

$$i_m = \frac{v_m}{Z} \tag{7.25(b)}$$

where $$Z = \sqrt{R^2 + (X_C - X_L)^2} \tag{7.26}$$

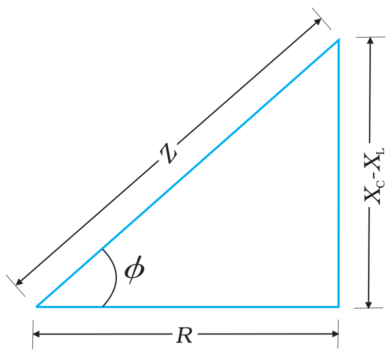

FIGURE 7.14 Impedance diagram.

Since phasor **I** is always parallel to phasor $\mathbf{V_R}$, the phase angle ϕ is the angle between $\mathbf{V_R}$ and **V** and can be determined from Fig. 7.14:

$$\tan \phi = \frac{v_{Cm} - v_{Lm}}{v_{Rm}}$$

Using Eq. (7.22), we have

$$\tan \phi = \frac{X_C - X_L}{R} \tag{7.27}$$

Equations (7.26) and (7.27) are graphically shown in Fig. (7.14). This is called *Impedance diagram* which is a right-triangle with Z as its hypotenuse.

Equation 7.25(a) gives the amplitude of the current and Eq. (7.27) gives the phase angle. With these, Eq. (7.21) is completely specified.

If $X_C > X_L$, ϕ is positive and the circuit is predominantly capacitive. Consequently, the current in the circuit leads the source voltage. If $X_C < X_L$, ϕ is negative and the circuit is predominantly inductive. Consequently, the current in the circuit lags the source voltage.

Figure 7.15 shows the phasor diagram and variation of v and i with ωt for the case $X_C > X_L$.

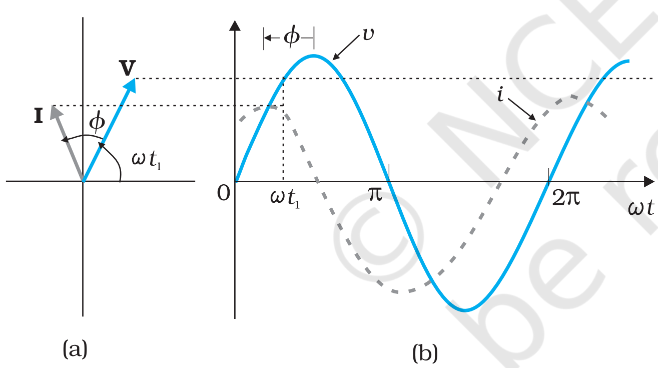

FIGURE 7.15 (a) Phasor diagram of **V** and **I**. (b) Graphs of v and i versus ωt for a series LCR circuit where $X_C > X_L$.

Thus, we have obtained the amplitude and phase of current for an LCR series circuit using the technique of phasors. But this method of analysing ac circuits suffers from certain disadvantages. First, the phasor diagram say nothing about the initial condition. One can take any arbitrary value of t (say, t_1, as done throughout this chapter) and draw different phasors which show the relative angle between different phasors. The solution so obtained is called the *steady-state solution*. This is not a general solution. Additionally, we do have a *transient solution* which exists even for $v = 0$. The general solution is the sum of the transient solution and the steady-state solution. After a sufficiently long time, the effects of the transient solution die out and the behaviour of the circuit is described by the steady-state solution.

7.6.2 Analytical solution

The voltage equation for the circuit is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + \frac{q}{C} = v$$

$$= v_m \sin \omega t$$

We know that $i = \mathrm{d}q/\mathrm{d}t$. Therefore, $\mathrm{d}i/\mathrm{d}t = \mathrm{d}^2q/\mathrm{d}t^2$. Thus, in terms of q, the voltage equation becomes

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=v_m \sin \omega t \qquad (7.28)$$

This is like the equation for a forced, damped oscillator, [see Eq. {14.37(b)} in Class XI Physics Textbook]. Let us assume a solution

$$q = q_m \sin(\omega t + \theta) \qquad [7.29(a)]$$

so that $\frac{dq}{dt}=q_m\omega\cos(\omega t+\theta)$ [7.29(b)]

and $\frac{d^2q}{dt^2}=-q_m\omega^2\sin(\omega t+\theta)$ [7.29(c)]

Substituting these values in Eq. (7.28), we get

$$q_m\omega\left[R\cos(\omega t+\theta)+(X_C-X_L)\sin(\omega t+\theta)\right]=v_m\sin\omega t \qquad (7.30)$$

where we have used the relation $X_c = 1/\omega C$, $X_L = \omega L$. Multiplying and dividing Eq. (7.30) by $Z=\sqrt{R^2+\left(X_c-X_L\right)^2}$, we have

$$q_m\omega Z\left[\frac{R}{Z}\cos(\omega t+\theta)+\frac{(X_C-X_L)}{Z}\sin(\omega t+\theta)\right]=v_m\sin\omega t \qquad (7.31)$$

Now, let $\frac{R}{Z}=\cos\phi$

and $\frac{(X_C-X_L)}{Z}=\sin\phi$

so that $\phi=\tan^{-1}\frac{X_C-X_L}{R}$ (7.32)

Substituting this in Eq. (7.31) and simplifying, we get:

$$q_m\omega Z\cos(\omega t+\theta-\phi)=v_m\sin\omega t \qquad (7.33)$$

Comparing the two sides of this equation, we see that

$$v_m = q_m\,\omega Z = i_m Z$$

where

$$i_m = q_m\,\omega \qquad [7.33(a)]$$

and $\theta-\phi=-\frac{\pi}{2}$ or $\theta=-\frac{\pi}{2}+\phi$ [7.33(b)]

Therefore, the current in the circuit is

$$i=\frac{dq}{dt}=q_m\,\omega\cos(\omega t+\theta)$$

$$= i_m \cos(\omega t + \theta)$$

or $i = i_m\sin(\omega t + \phi)$ (7.34)

where $i_m=\frac{v_m}{Z}=\frac{v_m}{\sqrt{R^2+(X_C-X_L)^2}}$ [7.34(a)]

and $\phi=\tan^{-1}\frac{X_C-X_L}{R}$

Thus, the analytical solution for the amplitude and phase of the current in the circuit agrees with that obtained by the technique of phasors.

7.6.3 Resonance

An interesting characteristic of the series *RLC* circuit is the phenomenon of resonance. The phenomenon of resonance is common among systems that have a tendency to oscillate at a particular frequency. This frequency is called the system's *natural frequency*. If such a system is driven by an energy source at a frequency that is near the natural frequency, the amplitude of oscillation is found to be large. A familiar example of this phenomenon is a child on a swing. The swing has a natural frequency for swinging back and forth like a pendulum. If the child pulls on the rope at regular intervals and the frequency of the pulls is almost the same as the frequency of swinging, the amplitude of the swinging will be large (Chapter 14, Class XI).

For an *RLC* circuit driven with voltage of amplitude v_m and frequency ω, we found that the current amplitude is given by

$$i_m = \frac{v_m}{Z} = \frac{v_m}{\sqrt{R^2 + (X_C - X_L)^2}}$$

with $X_c = 1/\omega C$ and $X_L = \omega L$. So if ω is varied, then at a particular frequency ω_0, $X_c = X_L$, and the impedance is minimum $\left(Z = \sqrt{R^2 + 0^2} = R\right)$. This frequency is called the *resonant frequency*:

$$X_c = X_L \text{ or } \frac{1}{\omega_0 C} = \omega_0 L$$

$$\text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad (7.35)$$

At resonant frequency, the current amplitude is maximum; $i_m = v_m/R$.

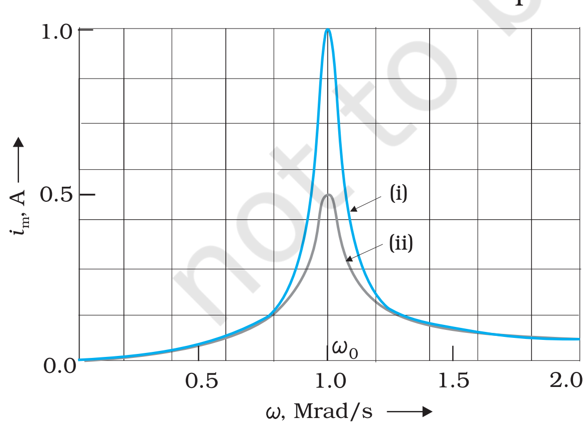

FIGURE 7.16 Variation of i_m with ω for two cases: (i) R = 100 Ω, (ii) R = 200 Ω, L = 1.00 mH.

Figure 7.16 shows the variation of i_m with ω in a *RLC* series circuit with L = 1.00 mH, C = 1.00 nF for two values of R: (i) R = 100 Ω and (ii) R = 200 Ω. For the source applied v_m = 100 V. ω_0 for this case is $\frac{1}{\sqrt{LC}} = 1.00\times10^6$ rad/s.

We see that the current amplitude is maximum at the resonant frequency. Since $i_m = v_m / R$ at resonance, the current amplitude for case (i) is twice to that for case (ii).

Resonant circuits have a variety of applications, for example, in the tuning mechanism of a radio or a TV set. The antenna of a radio accepts signals from many broadcasting stations. The signals picked up in the antenna acts as a source in the tuning circuit of the radio, so the circuit can be driven at many frequencies.