Thus, the analytical solution for the amplitude and phase of the current in the circuit agrees with that obtained by the technique of phasors.

7.6.3 Resonance

An interesting characteristic of the series *RLC* circuit is the phenomenon of resonance. The phenomenon of resonance is common among systems that have a tendency to oscillate at a particular frequency. This frequency is called the system's *natural frequency*. If such a system is driven by an energy source at a frequency that is near the natural frequency, the amplitude of oscillation is found to be large. A familiar example of this phenomenon is a child on a swing. The swing has a natural frequency for swinging back and forth like a pendulum. If the child pulls on the rope at regular intervals and the frequency of the pulls is almost the same as the frequency of swinging, the amplitude of the swinging will be large (Chapter 14, Class XI).

For an *RLC* circuit driven with voltage of amplitude v_m and frequency ω, we found that the current amplitude is given by

$$i_m = \frac{v_m}{Z} = \frac{v_m}{\sqrt{R^2 + (X_C - X_L)^2}}$$

with $X_c = 1/\omega C$ and $X_L = \omega L$. So if ω is varied, then at a particular frequency ω_0, $X_c = X_L$, and the impedance is minimum $\left(Z = \sqrt{R^2 + 0^2} = R\right)$. This frequency is called the *resonant frequency*:

$$X_c = X_L \text{ or } \frac{1}{\omega_0 C} = \omega_0 L$$

$$\text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}} \tag{7.35}$$

At resonant frequency, the current amplitude is maximum; $i_m = v_m/R$.

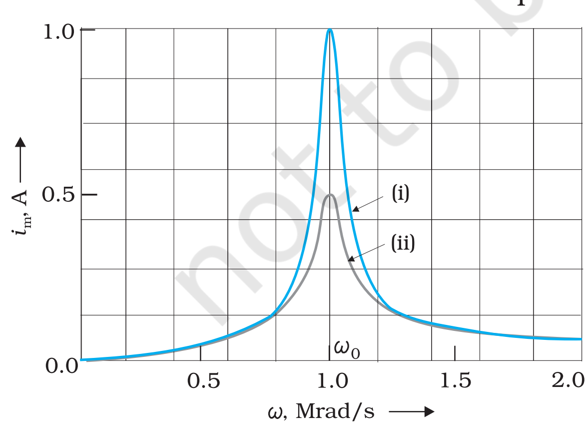

FIGURE 7.16 Variation of i_m with ω for two cases: (i) R = 100 Ω, (ii) R = 200 Ω, L = 1.00 mH.

Figure 7.16 shows the variation of i_m with ω in a *RLC* series circuit with L = 1.00 mH, C = 1.00 nF for two values of R: (i) R = 100 Ω and (ii) R = 200 Ω. For the source applied v_m = 100 V. ω_0 for this case is $\frac{1}{\sqrt{LC}} = 1.00 \times 10^6$ rad/s.

We see that the current amplitude is maximum at the resonant frequency. Since $i_m = v_m / R$ at resonance, the current amplitude for case (i) is twice to that for case (ii).

Resonant circuits have a variety of applications, for example, in the tuning mechanism of a radio or a TV set. The antenna of a radio accepts signals from many broadcasting stations. The signals picked up in the antenna acts as a source in the tuning circuit of the radio, so the circuit can be driven at many frequencies.

But to hear one particular radio station, we tune the radio. In tuning, we vary the capacitance of a capacitor in the tuning circuit such that the resonant frequency of the circuit becomes nearly equal to the frequency of the radio signal received. When this happens, the amplitude of the current with the frequency of the signal of the particular radio station in the circuit is maximum.

It is important to note that resonance phenomenon is exhibited by a circuit only if both L and C are present in the circuit. Only then do the voltages across L and C cancel each other (both being out of phase) and the current amplitude is v_m/R*, the total source voltage appearing across R. This means that we cannot have resonance in a RL or RC circuit.*

Sharpness of resonance

The amplitude of the current in the series *LCR* circuit is given by

$$i_m = \frac{v_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and is maximum when $\omega = \omega_0 = 1/\sqrt{LC}$. The maximum value is $i_m^{\max} = v_m / R$.

For values of ω other than ω_0, the amplitude of the current is less than the maximum value. Suppose we choose a value of ω for which the current amplitude is $1/\sqrt{2}$ times its maximum value. At this value, the power dissipated by the circuit becomes half. From the curve in Fig. (7.16), we see that there are two such values of ω, say, ω_1 and ω_2, one greater and the other smaller than ω_0 and symmetrical about ω_0. We may write

$$\omega_1 = \omega_0 + \Delta\omega$$

$$\omega_2 = \omega_0 - \Delta\omega$$

The difference $\omega_1 - \omega_2 = 2\Delta\omega$ is often called the *bandwidth* of the circuit. The quantity $(\omega_0 / 2\Delta\omega)$ is regarded as a measure of the sharpness of resonance. The smaller the $\Delta\omega$, the sharper or narrower is the resonance. To get an expression for $\Delta\omega$, we note that the current amplitude i_m is $(1/\sqrt{2})\, i_m^{\max}$ for $\omega_1 = \omega_0 + \Delta\omega$. Therefore,

$$\text{at } \omega_1, \quad i_m = \frac{v_m}{\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}}$$

$$= \frac{i_m^{\max}}{\sqrt{2}} = \frac{v_m}{R\sqrt{2}}$$

or $$\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2} = R\sqrt{2}$$

or $$R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2 = 2R^2$$

$$\omega_1 L - \frac{1}{\omega_1 C} = R$$

which may be written as,

$$(\omega_0 + \Delta\omega) L - \frac{1}{(\omega_0 + \Delta\omega) C} = R$$

$$\omega_0 L \left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0 C \left(1 + \frac{\Delta\omega}{\omega_0}\right)} = R$$

Using $\omega_0^2 = \frac{1}{LC}$ in the second term on the left hand side, we get

$$\omega_0 L \left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{\omega_0 L}{\left(1 + \frac{\Delta\omega}{\omega_0}\right)} = R$$

We can approximate $\left(1 + \frac{\Delta\omega}{\omega_0}\right)^{-1}$ as $\left(1 - \frac{\Delta\omega}{\omega_0}\right)$ since $\frac{\Delta\omega}{\omega_0} << 1$. Therefore,

$$\omega_0 L \left(1 + \frac{\Delta\omega}{\omega_0}\right) - \omega_0 L \left(1 - \frac{\Delta\omega}{\omega_0}\right) = R$$

or $$\omega_0 L \frac{2\Delta\omega}{\omega_0} = R$$

$$\Delta\omega = \frac{R}{2L} \qquad [7.36(a)]$$

The sharpness of resonance is given by,

$$\frac{\omega_0}{2\Delta\omega} = \frac{\omega_0 L}{R} \qquad [7.36(b)]$$

The ratio $\frac{\omega_0 L}{R}$ is also called the *quality factor*, Q of the circuit.

$$Q = \frac{\omega_0 L}{R} \qquad [7.36(c)]$$

From Eqs. [7.36 (b)] and [7.36 (c)], we see that $2\Delta\omega = \frac{\omega_0}{Q}$. So, larger the value of Q, the smaller is the value of $2\Delta\omega$ or the bandwidth and sharper is the resonance. Using $\omega_0^2 = 1/LC$, Eq. [7.36(c)] can be equivalently expressed as $Q = 1/\omega_0 CR$.

We see from Fig. 7.15, that if the resonance is less sharp, not only is the maximum current less, the circuit is close to resonance for a larger range $\Delta\omega$ of frequencies and the tuning of the circuit will not be good. So, less sharp the resonance, less is the selectivity of the circuit or vice versa. From Eq. (7.36), we see that if quality factor is large, i.e., R is low or L is large, the circuit is more selective.

Example 7.6 A resistor of 200 Ω and a capacitor of 15.0 µF are connected in series to a 220 V, 50 Hz ac source. (a) Calculate the current in the circuit; (b) Calculate the voltage (rms) across the resistor and the capacitor. Is the algebraic sum of these voltages more than the source voltage? If yes, resolve the paradox.

Solution

Given

$R = 200\,\Omega,\ C = 15.0\,\mu\text{F} = 15.0\times10^{-6}\,\text{F}$

$V = 220\,\text{V},\ \nu = 50\,\text{Hz}$

(a) In order to calculate the current, we need the impedance of the circuit. It is

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + (2\pi\nu C)^{-2}}$$

$$= \sqrt{(200\,\Omega)^2 + (2\times3.14\times50\times15.0\times10^{-6}\,\text{F})^{-2}}$$

$$= \sqrt{(200\,\Omega)^2 + (212.3\,\Omega)^2}$$

$$= 291.67\,\Omega$$

Therefore, the current in the circuit is

$$I = \frac{V}{Z} = \frac{220\,\text{V}}{291.5\,\Omega} = 0.755\,\text{A}$$

(b) Since the current is the same throughout the circuit, we have

$$V_R = I\,R = (0.755\,\text{A})(200\,\Omega) = 151\,\text{V}$$

$$V_C = I\,X_C = (0.755\,\text{A})(212.3\,\Omega) = 160.3\,\text{V}$$

The algebraic sum of the two voltages, V_R and V_C is 311.3 V which is more than the source voltage of 220 V. How to resolve this paradox? As you have learnt in the text, the two voltages are not in the same phase. Therefore, *they cannot be added like ordinary numbers*. The two voltages are out of phase by ninety degrees. Therefore, the total of these voltages must be obtained using the Pythagorean theorem:

$$V_{R+C} = \sqrt{V_R^2 + V_C^2}$$

$$= 220\,\text{V}$$

Thus, if the phase difference between two voltages is properly taken into account, the total voltage across the resistor and the capacitor is equal to the voltage of the source.

EXAMPLE 7.6

7.7 Power in AC Circuit: The Power Factor

We have seen that a voltage $v = v_m \sin\omega t$ applied to a series *RLC* circuit drives a current in the circuit given by $i = i_m \sin(\omega t + \phi)$ where

$$i_m = \frac{v_m}{Z} \text{ and } \phi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$

Therefore, the instantaneous power *p* supplied by the source is

$$p = v\,i = (v_m \sin\omega t) \times [i_m \sin(\omega t + \phi)]$$

$$= \frac{v_m i_m}{2}[\cos\phi - \cos(2\omega t + \phi)] \qquad (7.37)$$

The average power over a cycle is given by the average of the two terms in R.H.S. of Eq. (7.37). It is only the second term which is time-dependent. Its average is zero (the positive half of the cosine cancels the negative half). Therefore,

$$P = \frac{v_m i_m}{2}\cos\phi = \frac{v_m}{\sqrt{2}}\frac{i_m}{\sqrt{2}}\cos\phi$$

$$= V\,I\cos\phi \qquad [7.38(a)]$$

This can also be written as,

$$P = I^2 Z\cos\phi \qquad [7.38(b)]$$

So, the average power dissipated depends not only on the voltage and current but also on the cosine of the phase angle ϕ between them. The quantity $\cos\phi$ is called the *power factor*. Let us discuss the following cases:

Case (i) *Resistive circuit*: If the circuit contains only pure *R*, it is called *resistive*. In that case $\phi = 0$, $\cos\phi = 1$. There is maximum power dissipation.

Case (ii) *Purely inductive or capacitive circuit*: If the circuit contains only an inductor or capacitor, we know that the phase difference between voltage and current is $\pi/2$. Therefore, $\cos\phi = 0$, and no power is dissipated even though a current is flowing in the circuit. This current is sometimes referred to as *wattless current*.

Case (iii) *LCR series circuit*: In an *LCR* series circuit, power dissipated is given by Eq. (7.38) where $\phi = \tan^{-1}(X_c - X_L)/R$. So, ϕ may be non-zero in a *RL* or *RC* or *RCL* circuit. Even in such cases, power is dissipated only in the resistor.

Case (iv) *Power dissipated at resonance in LCR circuit*: At resonance $X_c - X_L = 0$, and $\phi = 0$. Therefore, $\cos\phi = 1$ and $P = I^2 Z = I^2 R$. That is, maximum power is dissipated in a circuit (through *R*) at resonance.

Example 7.7

Example 7.7 (a) For circuits used for transporting electric power, a low power factor implies large power loss in transmission. Explain.

(b) Power factor can often be improved by the use of a capacitor of appropriate capacitance in the circuit. Explain.

Solution (a) We know that $P = I\,V\cos\phi$ where $\cos\phi$ is the power factor. To supply a given power at a given voltage, if $\cos\phi$ is small, we have to increase current accordingly. But this will lead to large power loss (I^2R) in transmission.

(b)Suppose in a circuit, current I lags the voltage by an angle ϕ. Then power factor $\cos\phi = R/Z$.

We can improve the power factor (tending to 1) by making Z tend to R. Let us understand, with the help of a phasor diagram (Fig. 7.17) how this can be achieved. Let us resolve **I** into two components. $\mathbf{I}_p$ along

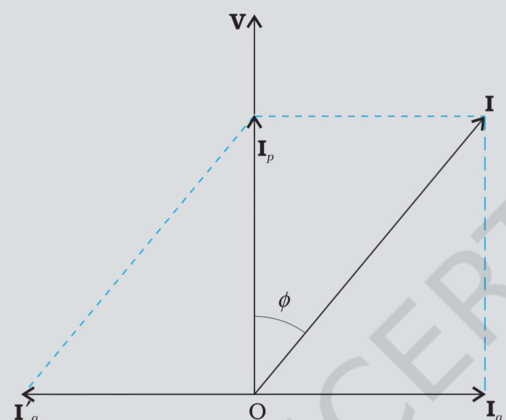

FIGURE 7.17

the applied voltage **V** and $\mathbf{I}_q$ perpendicular to the applied voltage. $\mathbf{I}_q$ as you have learnt in Section 7.7, is called the wattless component since corresponding to this component of current, there is no power loss. $\mathbf{I}_P$ is known as the power component because it is in phase with the voltage and corresponds to power loss in the circuit.

It's clear from this analysis that if we want to improve power factor, we must completely neutralize the lagging wattless current $\mathbf{I}_q$ by an equal leading wattless current $\mathbf{I}'_q$. This can be done by connecting a capacitor of appropriate value in parallel so that $\mathbf{I}_q$ and $\mathbf{I}'_q$ cancel each other and P is effectively $I_p\,V$.

EXAMPLE 7.7

Example 7.8 A sinusoidal voltage of peak value 283 V and frequency 50 Hz is applied to a series *LCR* circuit in which R = 3 Ω, L = 25.48 mH, and C = 796 μF. Find (a) the impedance of the circuit; (b) the phase difference between the voltage across the source and the current; (c) the power dissipated in the circuit; and (d) the power factor.

Solution

(a) To find the impedance of the circuit, we first calculate X_L and X_C.

$$X_L = 2\,\pi\nu L$$

$$= 2 \times 3.14 \times 50 \times 25.48 \times 10^{-3}\ \Omega = 8\ \Omega$$

$$X_C = \frac{1}{2\pi\nu C}$$

EXAMPLE 7.8

$$= \frac{1}{2 \times 3.14 \times 50 \times 796 \times 10^{-6}} = 4\Omega$$

Therefore,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{3^2 + (8-4)^2}$$

$$= 5\ \Omega$$

(b) Phase difference, $\phi = \tan^{-1}\frac{X_C - X_L}{R}$

$$= \tan^{-1}\left(\frac{4-8}{3}\right) = -53.1°$$

Since ϕ is negative, the current in the circuit lags the voltage across the source.

(c) The power dissipated in the circuit is

$$P = I^2 R$$

Now, $I = \frac{i_m}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{283}{5}\right) = 40\text{A}$

Therefore, $P = (40\text{A})^2 \times 3\,\Omega = 4800\text{ W}$

(d) Power factor $= \cos\phi = \cos(-53.1°) = 0.6$

EXAMPLE 7.8

Example 7.9 Suppose the frequency of the source in the previous example can be varied. (a) What is the frequency of the source at which resonance occurs? (b) Calculate the impedance, the current, and the power dissipated at the resonant condition.

Solution

(a) The frequency at which the resonance occurs is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{25.48 \times 10^{-3} \times 796 \times 10^{-6}}}$$

$$= 222.1\text{ rad/s}$$

$$\nu_r = \frac{\omega_0}{2\pi} = \frac{221.1}{2 \times 3.14}\text{Hz} = 35.4\text{Hz}$$

(b) The impedance Z at resonant condition is equal to the resistance:

$$Z = R = 3\,\Omega$$

The rms current at resonance is

$$= \frac{V}{Z} = \frac{V}{R} = \left(\frac{283}{\sqrt{2}}\right)\frac{1}{3} = 66.7\text{A}$$

The power dissipated at resonance is

$$P = I^2 \times R = (66.7)^2 \times 3 = 13.35\text{ kW}$$

You can see that in the present case, power dissipated at resonance is more than the power dissipated in Example 7.8.

EXAMPLE 7.9

Example 7.10 At an airport, a person is made to walk through the doorway of a metal detector, for security reasons. If she/he is carrying anything made of metal, the metal detector emits a sound. On what principle does this detector work?

Solution The metal detector works on the principle of resonance in ac circuits. When you walk through a metal detector, you are, in fact, walking through a coil of many turns. The coil is connected to a capacitor tuned so that the circuit is in resonance. When you walk through with metal in your pocket, the impedance of the circuit changes – resulting in significant change in current in the circuit. This change in current is detected and the electronic circuitry causes a sound to be emitted as an alarm.

EXAMPLE 7.10

7.8 LC OSCILLATIONS

We know that a capacitor and an inductor can store electrical and magnetic energy, respectively. When a capacitor (initially charged) is connected to an inductor, the charge on the capacitor and the current in the circuit exhibit the phenomenon of electrical oscillations similar to oscillations in mechanical systems (Chapter 14, Class XI).

Let a capacitor be charged q_m (at $t = 0$) and connected to an inductor as shown in Fig. 7.18.

The moment the circuit is completed, the charge on the capacitor starts decreasing, giving rise to current in the circuit. Let q and i be the charge and current in the circuit at time t. Since di/dt is positive, the induced emf in L will have polarity as shown, i.e., $v_b < v_a$. According to Kirchhoff's loop rule,

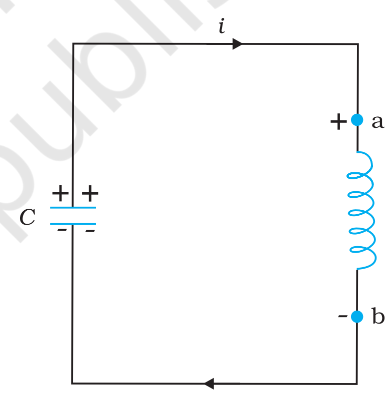

FIGURE 7.18 At the instant shown, the current is increasing; so the polarity of induced emf in the inductor is as shown.

$$\frac{q}{C} - L\frac{di}{dt} = 0 \qquad (7.39)$$

$i = -(dq/dt)$ in the present case (as q decreases, i increases). Therefore, Eq. (7.39) becomes:

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \qquad (7.40)$$

This equation has the form $\frac{d^2x}{dt^2} + \omega_0^2 x = 0$ for a simple harmonic oscillator. The charge on the capacitor, therefore, oscillates with a natural frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad (7.41)$$

and varies sinusoidally with time as

$$q = q_m \cos(\omega_0 t + \phi) \qquad (7.42)$$

where q_m is the maximum value of q and ϕ is a phase constant. Since $q = q_m$ at $t = 0$, we have $\cos\phi = 1$ or $\phi = 0$. Therefore, in the present case,