JEE problems on Integral of a given function:

Question 1: (JEE Main 2019)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}dx$, then the value of $(4I - 1)$ is

Sol:

$$I = \int_{-1}^{0} \frac{x \cdot 0}{2+0}dx + \int_{0}^{1} \frac{x \cdot 0}{2+1}dx + \int_{1}^{\sqrt{2}} \frac{x \cdot 1}{2+0}dx + 0 = \frac{1}{4}$$

$$\Rightarrow 4I - 1 = 0$$

Question 2: (JEE Main 2019)

If $f(x) = \frac{2 - x\cos x}{2 + x\cos x}$ and $g(x) = \log_e x, (x > 0)$

then the value of integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} g(f(x))dx$ is :

(1) $\log_e 3$ (2) $\log_e 2$

(3) $\log_e e$ (4) $\log_e 1$

Sol:

$$g(f(x)) = \ell n(f(x)) = \ell n\left(\frac{2 - x.\cos x}{2 + x.\cos x}\right)$$

$$\therefore \quad I = \int_{-\pi/4}^{\pi/4} \ell n\left(\frac{2 - x.\cos x}{2 + x.\cos x}\right)dx$$

$$= \int_{0}^{\pi/4} \left(\ell n\left(\frac{2 - x.\cos x}{2 + x.\cos x}\right) + \ell n\left(\frac{2 + x.\cos x}{2 - x.\cos x}\right)\right)dx$$

$$= \int_{0}^{\pi/2} (0)dx = 0 = \log_e(1)$$

Question 3: (JEE Main 2020)

If $f(a + b + 1 - x) = f(x)$, for all x, where a and b are fixed positive real numbers, then

$\frac{1}{a+b}\int_a^b x(f(x)+f(x+1))dx$ is equal to :

(1) $\int_{a+1}^{b+1} f(x)dx$ (2) $\int_{a+1}^{b+1} f(x+1)dx$

(3) $\int_{a-1}^{b-1} f(x+1)dx$ (4) $\int_{a-1}^{b-1} f(x)dx$

Sol:

$f(x + 1) = f(a + b - x)$

$$I = \frac{1}{(a+b)}\int_a^b x\left(f(x)+f(x+1)\right)dx \quad .....(1)$$

$$I = \frac{1}{(a+b)}\int_a^b (a+b-x)\left(f(x+1)+f(x)\right)dx \quad ...(2)$$

from (1) and (2)

$$2I = \int_a^b \left(f(x)+f(x+1)\right)dx$$

$$2I = \int_a^b f\left(a+b-x\right)dx + \int_a^b f(x+1)dx$$

$$2I = 2\int_a^b f(x+1)dx \Rightarrow I = \int_a^b f(x+1)dx$$

$$= \int_{a+1}^{b+1} f(x)dx$$

Question 4: (JEE Main 2020)

Let [t] denote the greatest integer less than or equal to t. Then the value of $\int_1^2 |2x-[3x]|dx$ is_______.

Sol:

$3 < 3x < 6$

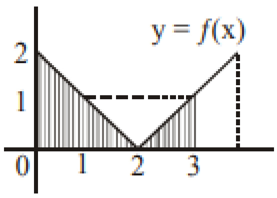
Take cases when $3 < 3x < 4$, $4 < 3x < 5$, $5 < 3x < 6$;

Now $\int_1^2 |2x-[3x]|dx$

$$= \int_1^{4/3}(3-2x)dx + \int_{4/3}^{5/3}(4-2x)\,dx + \int_{5/3}^{2}(5-2x)dx$$

$$= \frac{2}{9}+\frac{3}{9}+\frac{4}{9} = 1$$

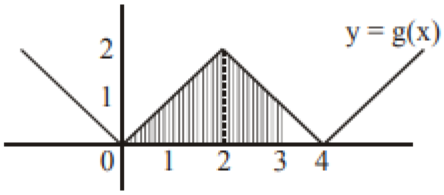
Question 5: (JEE Main 2020)



Let $f(x) = |x - 2|$ and $g(x) = f(f(x))$, $x \in [0, 4]$.

Then $\int_0^3 (g(x) - f(x))dx$ is equal to :

(1) $\frac{3}{2}$ (2) 0 (3) $\frac{1}{2}$ (4) 1



Sol:

$$\int_0^3 g(x) - f(x) = \int_0^3 \left||x-2|-2\right|dx - \int_0^3 |x-2|dx$$

$$= \left(\frac{1}{2}\times 2\times 2 + 1 + \frac{1}{2}\times 1\times 1\right) - \left(\frac{1}{2}\times 2\times 2 + \frac{1}{2}\times 1\times 1\right)$$

$$= \left(2+1+\frac{1}{2}\right) - \left(2+\frac{1}{2}\right) = 1$$