3. **Maximum and Minimum Values of Quadratic Expression**

(i) If $a > 0$, quadratic expression has least value at $x = b / 2a$. This least value is given by $4ac – b^2 / 4a = – D/4a$. But their is no greatest value.

(ii) If $a < 0$, quadratic expression has greatest value at $x = – b/2a$. This greatest value is given by $4ac – b^2 / 4a = – D/4a$. But their is no least value.

4. **Sign of Quadratic Expression**

(i) $a > 0$ and $D < 0$, so $f(x) > 0$ for all $x \in R$ i.e., $f(x)$ is positive for all real values of x.

(ii) $a < 0$ and $D < 0$, so $f(x) < 0$ for all $x \in R$ i.e., $f(x)$ is negative for all real values of x.

(iii) $a > 0$ and $D = 0$, so $f(x) \geq 0$ for all $x \in R$ i.e., $f(x)$ is positive for all real values of x except at vertex, where $f(x) = 0$.

(iv) $a < 0$ and $D = 0$, so $f(x) \leq 0$ for all $x \in R$ i.e., $f(x)$ is negative for all real values of x except at vertex, where $f(x) = 0$.

(v) $a > 0$ and $D > 0$
Let $f(x) = o$ have two real roots α and β ($\alpha < \beta$), then $f(x) > 0$ for $x \in (- \infty, \alpha) \cup (\beta,\infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.

(vi) $a < 0$ and $D > 0$
Let $f(x) = 0$ have two real roots α and β ($\alpha < \beta$). Then, $f(x) < 0$ for all $x \in (- \infty, \alpha) \cup (\beta,\infty)$ and $f(x) > 0$ for all
$x \in (\alpha, \beta)$. ,

5. **Intervals of Roots**

In some problems, we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.
Since, $a \neq 0$, we can take $f(x) = x^2 + b/a\ x + c/a$.

(i) Both the roots are positive i.e., they lie in $(0,\infty)$, if and only if roots are real, the sum of the roots as well as the product of the roots is positive.

$\alpha + \beta = -b/a > 0$ and $\alpha\beta = c/a > 0$ with $b^2 – 4ac \geq 0$

Similarly, both the roots are negative i.e., they lie in $(- \infty,0)$ ifF roots are real, the sum of the roots is negative and the product of the roots is positive.

i.e., $\alpha + \beta = -b/a < 0$ and $\alpha\beta = c/a > 0$ with $b^2 – 4ac \geq 0$

(ii) Both the roots are greater than a given number k, iFf the following conditions are satisfied

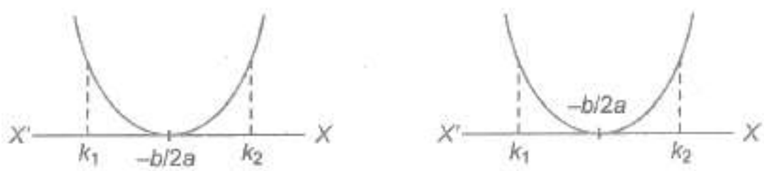
$D \geq 0$, $-b/2a > k$ and $f(k) > 0$

(iii) Both .the roots are less than a given number k, iff the following conditions are satisfied

$D \geq 0$, $-b/2a > k$ and $f(k) > 0$

(iv) Both the roots lie in a' given interval (k_1, k_2), iff the following conditions are satisfied

$D \geq 0, k_1 < -b/2a < k_2$ and $f(k_1) > 0$, $f(k_2) > 0$

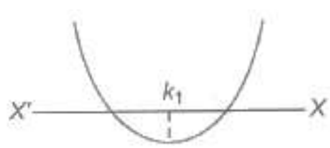


(v) Exactly one of the roots lie in a given interval (k_1, k_2), iff

$f(k_1)\, f(k_2) < 0$

(vi) A given number k lies between the roots iff $f(k) < O$. In particular, the roots of the equation will be of opposite sign, iff 0 lies between the roots.

$\Rightarrow f(0) < 0$



Wavy Curve Method

Let $f(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} (x - a_3)^{k_3} \dots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n}$

where $k_1, k_2, k_3, \ldots, k_n \in N$ and $a_1, a_2, a_3, \ldots, a_n$ are fixed natural numbers satisfying the condition.

$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$..

First we mark the numbers $a_1, a_2, a_3, \ldots, a_n$ on the real axis and the plus sign in the interval of the right of the largest of these numbers, i.e., on the right of a_n. If k_n is even, we put plus sign on the left of $a_n$and if k_n is odd, then we put minus sign on the left of a_n In the next interval we put a sign according to the following rule.

When passing through the point a_{n-1} the polynomial f(x) changes sign . if k_{n-1} is an odd number and the polynomial f(x) has same sign if k_{n-1} is an even number. Then, we consider the next interval and put a sign in it using the same rule.

Thus, we consider all the intervals. The solution of $f(x) > 0$ is the union of all interval in which we have put the plus sign and the solution of $f(x) < 0$ is the union of all intervals in which we have put the minus Sign.

Descarte's Rule of Signs

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in f(x) .

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in f(x).

Rational Algebraic In equations

(i) **Values of Rational Expression P(x)/Q(x)** for Real Values of x, where **P(x) and Q(x) are Quadratic Expressions** To find the values attained by rational expression of the form $a_1x^2 + b_1x + c_1 / a_2x^2 + b_2x + c_2$

for real values of x.

(a) Equate the given rational expression to y.
(b) Obtain a quadratic equation in x by simplifying the expression,
(c) Obtain the discriminant of the quadratic equation.
(d) Put discriminant ≥ 0 and solve the in equation for y. The values of y so obtained determines the set of values attained by the given rational expression.

(ii) **Solution of Rational Algebraic In equation** If P(x) and Q(x) are polynomial in x, then the in equation $P(x) / Q(x) > 0$,
$P(x) / Q(x) < 0$, $P(x) / Q(x) \geq 0$ and $P(x) / Q(x) \leq 0$ are known as rational algebraic in equations.

To solve these in equations we use the sign method as

(a) Obtain P(x) and Q(x).
(b) Factorize P(x) and Q(x) into linear factors.
(c) Make the coefficient of x positive in all factors.
(d) Obtain critical points by equating all factors to zero.
(e) Plot the critical points on the number line. If these are n critical points, they divide the number line into (n + 1) regions.
(f) In the right most region the expression P(x) / Q(x) bears positive sign and in other region the expression bears positive and negative signs depending on the exponents of the factors .

Lagrange's identity

If $a_1, a_2, a_3, b_1, b_2, b_3 \neq R$, then

$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$

Algebraic Interpretation of Rolle's Theorem

Let f (x) be a polynomial having α and β as its roots such that $\alpha < \beta$, $f(\alpha) = f(\beta) = 0$.Also, a polynomial function is everywhere continuous and differentiable, then there exist $\theta \in (\alpha, \beta)$ such that $f'(\theta) = 0$. Algebraically, we can say between any two zeros of a polynomial f(x) there is always a derivative f' (x) = 0.

Equation and In equation Containing Absolute Value

1. **Equation Containing Absolute Value**

By definition, $|x| = x$, if $x \geq 0$ OR $-x$, if $x < 0$

If $|f(x) + g(x)| = |f(x)| + g(x)|$, then it is equivalent to the system $f(x) . g(x) \geq 0$.

If $|f(x) - g(x)| = |f(x)| - g(x)|$, then it is equivalent to the system $f(x) . g(x) \leq 0$.

2.**In equation Containing Absolute Value**

(i) $|x| < a \Rightarrow -a < x < a \ (a > 0)$
(ii) $|x| \leq a \Rightarrow -a \leq x \leq a$
(iii) $|x| > a \Rightarrow x < -a$ or $x > a$
(iv) $|x| \geq a \Rightarrow$ x le; $-$ a or $x \geq a$

3. **Absolute Value of Real Number**

$|x| = -x, x < 0$ OR $+x, x \geq 0$

(i) $|xy| = |x||y|$
(ii) $|x / y| = |x| / |y|$
(iii) $|x|^2 = x^2$
(iv) $|x| \geq x$
(v) $|x + y| \leq |x| + |y|$
Equality hold when x and y same sign.
(vi) $|x - y| \geq ||x| - |y||$

Inequalities

Let a and b be real numbers. If a – b is negative, we say that a is less than b ($a < b$) and if a – b is positive, then a is greater than b ($a > b$).

Important Points to be Remembered

(i) If $a > b$ and $b > c$, then $a > c$. Generally, if $a_1 > a_2$, $a_2 > a_3$,…., $a_{n-1} > a_n$, then $a_1 > a_n$.

(ii) If $a > b$, then $a \pm c > b \pm c, \forall c \in R$

(iii) (a) If $a > b$ and $m > 0$, $am > bm, \frac{a}{m} > \frac{b}{m}$

(b) If $a > b$ and $m < 0$, $bm < am, \frac{b}{m} < \frac{a}{m}$

(iv) If $a > b > 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} < \frac{1}{b}$

(v) If $a < b < 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} > \frac{1}{b}$

(vi) If $a < 0 < b$, then

(a) $a^2 > b^2$, if $|a| > |b|$

(b) $a^2 < b^2$, if $|a| < |b|$

(vii) If $a < x < b$ and a, b are positive real numbers then $a^2 < x^2 < b^2$

(viii) If $a < x < b$ and a is negative number and b is positive number, then

(a) $0 \le x^2 < b^2$, if $|b| > |a|$

(b) $0 \le x^2 \le b^2$, if $|a| > |b|$

(ix) If $\frac{a}{b} > 0$, then

(a) $a > 0$, if $b > 0$

(b) $a < 0$, if $b < 0$

(x) If $a_i > b_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 a_2 a_3 \ldots a_n > b_1 b_2 b_3 \ldots b_n$$

(xi) If $|x| < a$ and

(a) if a is positive, then $-a < x < a$.

(b) if a is negative, then $x \in \phi$

(xii) If $a_i > b_i$, where $i = 1, 2, 3, \ldots, n$, then

$a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$

(xiii) If $0 < a < 1$ and n is a positive rational number, then

(a) $0 < a^n < 1$ (b) $a^{-n} > 1$

Important Inequality

1. **Arithmetico-Geometric and Harmonic Mean Inequality**

(i) If a, b > 0 and a ≠ b, then

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2}{(1/a)+(1/b)}$$

(ii) if $a_i > 0$, where i = 1,2,3,…,n, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

(iii) If $a_1, a_2, \ldots, a_n$ are n positive real numbers and $m_1, m_2, \ldots, m_n$ are n positive rational numbers, then

$$\frac{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n}{m_1 + m_2 + \ldots + m_n} > (a_1^{m_1} \cdot a_2^{m_2} \cdot \ldots \cdot a_n^{m_n})^{\frac{1}{m_1 + m_2 + \ldots + m_n}}$$

i.e., Weighted AM > Weighted GM

(iv) If $a_1, a_2, \ldots, a_n$ are n positive distinct real numbers, then

(a) $$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m \text{ if } m < 0 \text{ or } m > 1$$

(b) $$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m, \text{ if } 0 < m < 1$$

(c) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are rational numbers and M is a rational number, then

$$\frac{b_1 a_1^m + b_2 a_2^m + \ldots + b_n a_n^m}{b_1 + b_2 + \ldots + b_n} > \left(\frac{b_1 a_1 + b_2 a_2 + \ldots + b_n a_n}{b_1 + b_2 + \ldots + b_n}\right)^m, \text{if } 0 < m < 1$$

$$\frac{b_1 a_1^m + b_2 a_2^m + \ldots + b_n a_n^m}{b_1 + b_2 + \ldots + b_n} < \left(\frac{b_1 a_1 + b_2 a_2 + \ldots + b_n a_n}{b_1 + b_2 + \ldots + b_n}\right)^m,$$

(d) $\text{if } 0 < m < 1$

(v) If $a_1, a_2, a_3, \ldots, a_n$ are distinct positive real numbers and p, ,q, r are natural numbers, then

$$\frac{a_1^{p+q+r} + a_2^{p+q+r} + \ldots + a_n^{p+q+r}}{n} > \left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)$$
$$\left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)\left(\frac{a_1^r + a_2^r + \ldots + a_n^r}{n}\right)$$

2. **Cauchy – Schwartz's inequality**

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots, a_n^2) * (b_1^2 + b_2^2 + \ldots, b_n^2)$

Equality holds, iff $a_1 / b_1 = a_2 / b_2 = a_n / b_n$

3. **Tchebychef's Inequality**

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

(i) If $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

(ii) If If $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

4. **Weierstrass Inequality**

(i) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then for $n \geq 2$

$(1 + a_1)(1 + a_2) \ldots (1 + a_n) > 1 + a_1 + a_2 + \ldots + a_n$

(ii) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then

$(1 - a_1)(1 - a_2) \ldots (1 - a_n) > 1 - a_1 - a_2 - \ldots - a_n$

5. **Logarithm Inequality**

(i) (a) When $y > 1$ and $\log_y x > z \Rightarrow x > y^z$

(b) When $y > 1$ and $\log_y x < z \Rightarrow 0 < x < y^z$

(ii) (a) When $0 < y < 1$ and $\log_y x > z \Rightarrow 0 < x < y^z$

(b) hen $0 < y < 1$ and $\log_y x < z \Rightarrow x > y^z$

Application of Inequalities to Find the Greatest and Least Values

(i) If $x_l, x_2, \ldots, x_n$ are n positive variables such that $x_l + x_2 + \ldots + x_n = c$ (constant), then the product $x_l * x_2 * \ldots * x_n$ is greatest when $x_1 = x_2 = \ldots = x_n = c/n$ and the greatest value is $(c/n)^n$.

(ii) If $x_l, x_2, \ldots, x_n$ are positive variables such that $x_l, x_2, \ldots, x_n = c$ (constant), then the sum $x_l + x_2 + \ldots + x_n$ is least when $x_1 = x_2 = \ldots = x_n = c^{1/n}$ and the least value of the sum is $n\,(c^{1/n})$.

(iii) If $x_l, x_2, \ldots, x_n$ are variables and $m_l, m_2, \ldots, m_n$ are positive real number such that $x_l + x_2 + \ldots + x_n = c$ (constant), then $x_l^{m_l} * x_2^{m_2} * \ldots * x_n^{m_n}$ is greatest, when

$x_l / m_l = x_2 / m_2 = \ldots = x_n / m_n$

$= x_l + x_2 + \ldots + x_n / m_l + m_2 + \ldots + m_n$