6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

7. Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

12. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

13. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.

17. Show that the vectors $2\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}$ form the vertices of a right angled triangle.

18. If $\vec{a}$ is a nonzero vector of magnitude '$\boldsymbol{a}$' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $\boldsymbol{a} = |\lambda|$ (D) $\boldsymbol{a} = 1/|\lambda|$

10.6.3 *Vector (or cross) product of two vectors*

In Section 10.2, we have discussed on the three dimensional right handed rectangular coordinate system. In this system, when the positive ***x***-axis is rotated counterclockwise into the positive ***y***-axis, a right handed (standard) screw would advance in the direction of the positive ***z***-axis (Fig 10.22(i)).

In a right handed coordinate system, the thumb of the right hand points in the direction of the positive ***z***-axis when the fingers are curled in the direction away from the positive ***x***-axis toward the positive ***y***-axis (Fig 10.22(ii)).

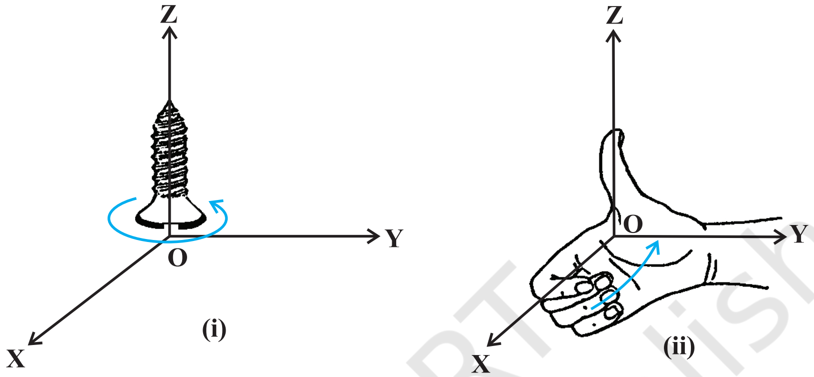

Fig 10.22 (i), (ii)

Definition 3 The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$, is denoted by $\vec{a} \times \vec{b}$ and defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (Fig 10.23). i.e., the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{\boldsymbol{n}}$.

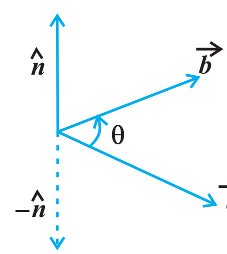

Fig 10.23

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a} \times \vec{b} = \vec{0}$.

Observations

1. $\vec{a} \times \vec{b}$ is a vector.

2. Let $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ be two nonzero vectors. Then $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,

$$\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$$

In particular, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times (-\vec{a}) = \vec{0}$, since in the first situation, θ = 0 and in the second one, θ = π, making the value of sin θ to be 0.

3. If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$.

4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ (Fig 10.24), we have

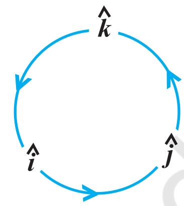

Fig 10.24

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

5. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

6. It is always true that the vector product is not commutative, as $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Indeed, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system, i.e., θ is traversed from $\vec{a}$ to $\vec{b}$, Fig 10.25 (i). While, $\vec{b} \times \vec{a} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}_1$, where $\vec{b}, \vec{a}$ and $\hat{n}_1$ form a right handed system i.e. θ is traversed from $\vec{b}$ to $\vec{a}$, Fig 10.25(ii).

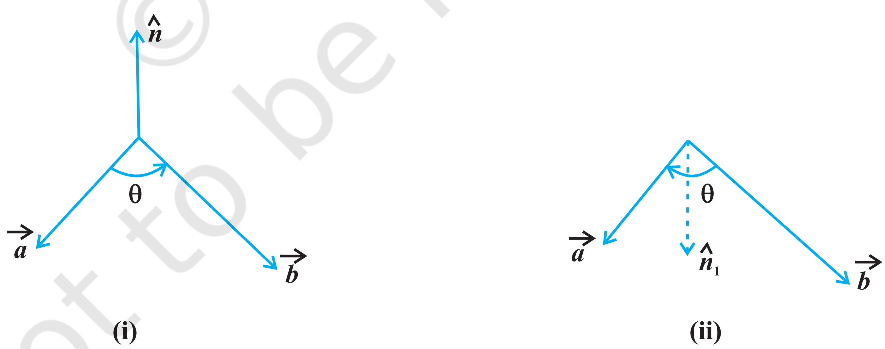

Fig 10.25 (i), (ii)

Thus, if we assume $\vec{a}$ and $\vec{b}$ to lie in the plane of the paper, then $\hat{n}$ and $\hat{n}_1$ both will be perpendicular to the plane of the paper. But, $\hat{n}$ being directed above the paper while $\hat{n}_1$ directed below the paper. i.e. $\hat{n}_1 = -\hat{n}$.

Hence $\qquad \vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$$= -|\vec{a}||\vec{b}|\sin\theta\,\hat{n}_1 = -\vec{b}\times\vec{a}$$

7. In view of the Observations 4 and 6, we have

$$\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}} = -\hat{\boldsymbol{k}},\quad \hat{\boldsymbol{k}}\times\hat{\boldsymbol{j}} = -\hat{\boldsymbol{i}} \text{ and } \hat{\boldsymbol{i}}\times\hat{\boldsymbol{k}} = -\hat{\boldsymbol{j}}.$$

8. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a}\times\vec{b}|$.

By definition of the area of a triangle, we have from Fig 10.26,

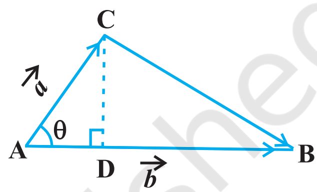

Fig 10.26

Area of triangle ABC $= \frac{1}{2}\text{AB}\cdot\text{CD}$.

But $\text{AB} = |\vec{b}|$ (as given), and $\text{CD} = |\vec{a}|\sin\theta$.

Thus, Area of triangle ABC $= \frac{1}{2}|\vec{b}||\vec{a}|\sin\theta = \frac{1}{2}|\vec{a}\times\vec{b}|$.

9. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$.

From Fig 10.27, we have

Area of parallelogram ABCD = AB. DE.

But $\text{AB} = |\vec{b}|$ (as given), and

$\text{DE} = |\vec{a}|\sin\theta$.

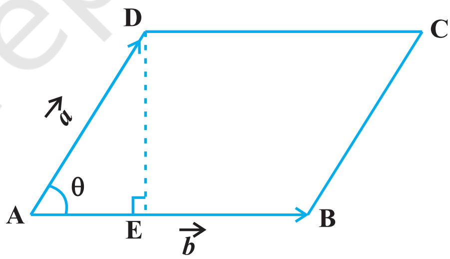

Fig 10.27

Thus,

Area of parallelogram ABCD $= |\vec{b}||\vec{a}|\sin\theta = |\vec{a}\times\vec{b}|$.

We now state two important properties of vector product.

Property 3 (Distributivity of vector product over addition): If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors and λ be a scalar, then

(i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(ii) $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

Let $\vec{a}$ and $\vec{b}$ be two vectors given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, respectively. Then their cross product may be given by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Explanation We have

$$\begin{aligned}
\vec{a}\times\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1b_1(\hat{i}\times\hat{i})+a_1b_2(\hat{i}\times\hat{j})+a_1b_3(\hat{i}\times\hat{k})+a_2b_1(\hat{j}\times\hat{i}) \\
&\quad + a_2b_2(\hat{j}\times\hat{j})+a_2b_3(\hat{j}\times\hat{k}) \\
&\quad + a_3b_1(\hat{k}\times\hat{i})+a_3b_2(\hat{k}\times\hat{j})+a_3b_3(\hat{k}\times\hat{k}) \qquad \text{(by Property 1)} \\
&= a_1b_2(\hat{i}\times\hat{j})-a_1b_3(\hat{k}\times\hat{i})-a_2b_1(\hat{i}\times\hat{j}) \\
&\quad + a_2b_3(\hat{j}\times\hat{k})+a_3b_1(\hat{k}\times\hat{i})-a_3b_2(\hat{j}\times\hat{k})
\end{aligned}$$

(as $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and $\hat{i}\times\hat{k}=-\hat{k}\times\hat{i}$, $\hat{j}\times\hat{i}=-\hat{i}\times\hat{j}$ and $\hat{k}\times\hat{j}=-\hat{j}\times\hat{k}$)

$$= a_1b_2\hat{k}-a_1b_3\hat{j}-a_2b_1\hat{k}+a_2b_3\hat{i}+a_3b_1\hat{j}-a_3b_2\hat{i}$$

(as $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$)

$$= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 22 Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$

Solution We have

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$= \hat{i}(-2-15)-(-4-9)\hat{j}+(10-3)\hat{k} = -17\hat{i}+13\hat{j}+7\hat{k}$$

Hence $|\vec{a}\times\vec{b}| = \sqrt{(-17)^2+(13)^2+(7)^2} = \sqrt{507}$

Example 23 Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

Solution We have $\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A vector which is perpendicular to both $\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{a}}-\vec{\boldsymbol{b}}$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k} \quad (=\vec{c},\text{ say})$$

Now $$|\vec{c}| = \sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}$$

Therefore, the required unit vector is

$$\frac{\vec{\boldsymbol{c}}}{|\vec{\boldsymbol{c}}|} = \frac{-1}{\sqrt{6}}\hat{\boldsymbol{i}}+\frac{2}{\sqrt{6}}\hat{\boldsymbol{j}}-\frac{1}{\sqrt{6}}\hat{\boldsymbol{k}}$$

Note There are two perpendicular directions to any plane. Thus, another unit vector perpendicular to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ will be $\frac{1}{\sqrt{6}}\hat{\boldsymbol{i}}-\frac{2}{\sqrt{6}}\hat{\boldsymbol{j}}+\frac{1}{\sqrt{6}}\hat{\boldsymbol{k}}$. But that will be a consequence of $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$.

Example 24 Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Solution We have $\overrightarrow{AB}=\hat{j}+2\hat{k}$ and $\overrightarrow{AC}=\hat{i}+2\hat{j}$. The area of the given triangle is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

Now, $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}$$

Therefore $$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{16+4+1}=\sqrt{21}$$

Thus, the required area is $\frac{1}{2}\sqrt{21}$

Example 25 Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Solution The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a} \times \vec{b}|$.

Now $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{i} + \hat{j} - 4\hat{k}$$

Therefore $$|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42}$$

and hence, the required area is $\sqrt{42}$.

EXERCISE 10.4

1. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.
2. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
3. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.
4. Show that
$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$$
5. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.
6. Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?
7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.
9. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $-\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Miscellaneous Examples

Example 26 Write all the unit vectors in XY-plane.

Solution Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane (Fig 10.28). Then, from the figure, we have $\boldsymbol{x} = \cos\theta$ and $\boldsymbol{y} = \sin\theta$ (since $|\vec{r}| = 1$). So, we may write the vector $\vec{r}$ as

$$\vec{r}\,(=\overrightarrow{OP}) = \cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}} \qquad \ldots (1)$$

Clearly,

$$|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

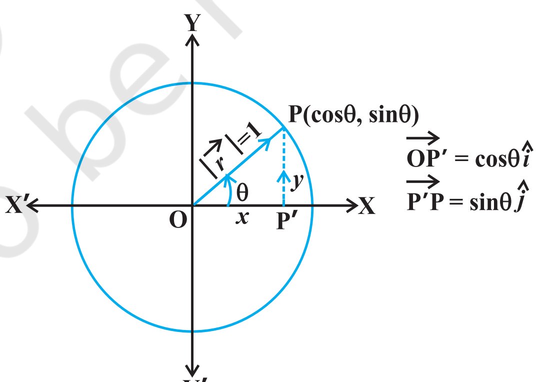

Fig 10.28

Also, as θ varies from 0 to 2π, the point P (Fig 10.28) traces the circle $x^2 + y^2 = 1$ counterclockwise, and this covers all possible directions. So, (1) gives every unit vector in the XY-plane.

Example 27 If $\hat{i}+\hat{j}+\hat{k},\ 2\hat{i}+5\hat{j},\ 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Solution Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$$

Therefore $$|\overrightarrow{AB}| = \sqrt{(1)^2+(4)^2+(-1)^2} = 3\sqrt{2}$$

Similarly $$\overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k} \text{ and } |\overrightarrow{CD}| = 6\sqrt{2}$$

Thus $$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$$

$$= \frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1$$

Since $0 \le \theta \le \pi$, it follows that $\theta = \pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Alternatively, $\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

Example 28 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution Given $\vec{a}\cdot(\vec{b}+\vec{c}) = 0,\ \vec{b}\cdot(\vec{c}+\vec{a}) = 0,\ \vec{c}\cdot(\vec{a}+\vec{b}) = 0.$

Now $$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot\vec{b}+\vec{b}\cdot(\vec{a}+\vec{c})$$

$$+\ \vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$$

$$= 9 + 16 + 25 = 50$$

Therefore $$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$$

Example 29 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=2$.

Solution Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have

$$\vec{a}+\vec{b}+\vec{c}=\vec{0} \quad = 0$$

or $$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} \quad = 0$$

Therefore $$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=-|\vec{a}|^2=-9 \qquad \text{... (1)}$$

Again, $$\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c}) \quad = 0$$

or $$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}=-|\vec{b}|^2=-16 \qquad \text{... (2)}$$

Similarly $$\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} \quad = -4. \qquad \text{... (3)}$$

Adding (1), (2) and (3), we have

$$2\,(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}) \quad = -29$$

or $$2\mu = -29, \text{ i.e., } \mu = \frac{-29}{2}$$

Example 30 If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, $\vec{\alpha}=3\hat{i}-\hat{j}$, $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Solution Let $\vec{\beta}_1=\lambda\vec{\alpha}$, λ is a scalar, i.e., $\vec{\beta}_1=3\lambda\hat{i}-\lambda\hat{j}$.

Now $$\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1 = (2-3\lambda)\hat{\boldsymbol{i}}+(1+\lambda)\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}}.$$

Now, since $\vec{\beta}_2$ is to be perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}\cdot\vec{\beta}_2=0$. i.e.,

$$3(2-3\lambda)-(1+\lambda) \quad = 0$$

or $$\lambda = \frac{1}{2}$$

Therefore $$\vec{\beta}_1 = \frac{3}{2}\hat{\boldsymbol{i}}-\frac{1}{2}\hat{\boldsymbol{j}} \quad \text{and} \quad \vec{\beta}_2=\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$

Miscellaneous Exercise on Chapter 10

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

2. Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

5. Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

8. Show that the points A (1, – 2, – 8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

10. The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\pm\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 15$.

13. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

14. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{c} \cdot \vec{d} = 15$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

15. Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Choose the correct answer in Exercises 16 to 19.

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

18. The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$ is

(A) 0 (B) –1 (C) 1 (D) 3

19. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Summary

- Position vector of a point P(*x*, *y*, *z*) is given as $\overrightarrow{OP}(=\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$, and its magnitude by $\sqrt{x^2+y^2+z^2}$.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude (*r*), direction ratios (*a*, *b*, *c*) and direction cosines (*l*, *m*, *n*) of any vector are related as:

$$l = \frac{a}{r},\quad m = \frac{b}{r},\quad n = \frac{c}{r}$$

- The vector sum of the three sides of a triangle taken in order is $\vec{0}$.
- The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar λ, changes the magnitude of the vector by the multiple $|\lambda|$, and keeps the direction same (or makes it opposite) according as the value of λ is positive (or negative).
- For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$.
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio ***m*** : ***n***

  (i) internally, is given by $\frac{n\vec{a} + m\vec{b}}{m+n}$.

  (ii) externally, is given by $\frac{m\vec{b} - n\vec{a}}{m-n}$.
- The scalar product of two given vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta .$$

  Also, when $\vec{a} \cdot \vec{b}$ is given, the angle 'θ' between the vectors $\vec{a}$ and $\vec{b}$ may be determined by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

- If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then their cross product is given as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

  where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Such that $\vec{a}, \vec{b}, \hat{n}$ form right handed system of coordinate axes.
- If we have two vectors $\vec{a}$ and $\vec{b}$, given in component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ any scalar,

then $\vec{a}+\vec{b} = (\boldsymbol{a}_1+\boldsymbol{b}_1)\hat{\boldsymbol{i}}+(\boldsymbol{a}_2+\boldsymbol{b}_2)\hat{\boldsymbol{j}}+(\boldsymbol{a}_3+\boldsymbol{b}_3)\hat{\boldsymbol{k}}$;

$\lambda\vec{a} = (\lambda\boldsymbol{a}_1)\hat{\boldsymbol{i}}+(\lambda\boldsymbol{a}_2)\hat{\boldsymbol{j}}+(\lambda\boldsymbol{a}_3)\hat{\boldsymbol{k}}$;

$\vec{a}\cdot\vec{b} = \boldsymbol{a}_1\boldsymbol{b}_1+\boldsymbol{a}_2\boldsymbol{b}_2+\boldsymbol{a}_3\boldsymbol{b}_3$;

and $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \boldsymbol{a}_1 & \boldsymbol{b}_1 & \boldsymbol{c}_1 \\ \boldsymbol{a}_2 & \boldsymbol{b}_2 & \boldsymbol{c}_2 \end{vmatrix}$.

Historical Note

The word ***vector*** has been derived from a Latin word ***vectus***, which means "to carry". The germinal ideas of modern vector theory date from around 1800 when Caspar Wessel (1745-1818) and Jean Robert Argand (1768-1822) described that how a complex number $\boldsymbol{a+ib}$ could be given a geometric interpretation with the help of a directed line segment in a coordinate plane. William Rowen Hamilton (1805-1865) an Irish mathematician was the first to use the term vector for a directed line segment in his book ***Lectures on Quaternions*** (1853). Hamilton's method of quaternions (an ordered set of four real numbers given as: $\boldsymbol{a}+\boldsymbol{b}\hat{\boldsymbol{i}}+\boldsymbol{c}\hat{\boldsymbol{j}}+\boldsymbol{d}\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$ following certain algebraic rules) was a solution to the problem of multiplying vectors in three dimensional space. Though, we must mention here that in practice, the idea of vector concept and their addition was known much earlier ever since the time of Aristotle (384-322 B.C.), a Greek philosopher, and pupil of Plato (427-348 B.C.). That time it was supposed to be known that the combined action of two or more forces could be seen by adding them according to parallelogram law. The correct law for the composition of forces, that forces add vectorially, had been discovered in the case of perpendicular forces by Stevin-Simon (1548-1620). In 1586 A.D., he analysed the principle of geometric addition of forces in his treatise ***DeBeghinselen der Weeghconst*** ("Principles of the Art of Weighing"), which caused a major breakthrough in the development of mechanics. But it took another 200 years for the general concept of vectors to form.

In the 1880, Josaih Willard Gibbs (1839-1903), an American physicist and mathematician, and Oliver Heaviside (1850-1925), an English engineer, created what we now know as ***vector analysis***, essentially by separating the real (***scalar***) part of quaternion from its imaginary (***vector***) part. In 1881 and 1884, Gibbs printed a treatise entitled ***Element of Vector Analysis***. This book gave a systematic and concise account of vectors. However, much of the credit for demonstrating the applications of vectors is due to the D. Heaviside and P.G. Tait (1831-1901) who contributed significantly to this subject.