Inequalities

Let a and b be real numbers. If $a - b$ is negative, we say that a is less than b ($a < b$) and if $a - b$ is positive, then a is greater than b ($a > b$).

Important Points to be Remembered

(i) If $a > b$ and $b > c$, then $a > c$. Generally, if $a_1 > a_2$, $a_2 > a_3, \dots, a_{n-1} > a_n$, then $a_1 > a_n$.

(ii) If $a > b$, then $a \pm c > b \pm c, \forall c \in R$

(iii) (a) If $a > b$ and $m > 0$, $am > bm, \frac{a}{m} > \frac{b}{m}$

(b) If $a > b$ and $m < 0$, $bm < am, \frac{b}{m} < \frac{a}{m}$

(iv) If $a > b > 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} < \frac{1}{b}$

(v) If $a < b < 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} > \frac{1}{b}$

(vi) If $a < 0 < b$, then

(a) $a^2 > b^2$, if $|a| > |b|$

(b) $a^2 < b^2$, if $|a| < |b|$

(vii) If $a < x < b$ and a, b are positive real numbers then $a^2 < x^2 < b^2$

(viii) If $a < x < b$ and a is negative number and b is positive number, then

(a) $0 \le x^2 < b^2$, if $|b| > |a|$

(b) $0 \le x^2 \le b^2$, if $|a| > |b|$

(ix) If $\frac{a}{b} > 0$, then

(a) $a > 0$, if $b > 0$

(b) $a < 0$, if $b < 0$

(x) If $a_i > b_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 a_2 a_3 \ldots a_n > b_1 b_2 b_3 \ldots b_n$$

(xi) If $|x| < a$ and

(a) if a is positive, then $-a < x < a$.

(b) if a is negative, then $x \in \phi$

(xii) If $a_i > b_i$, where $i = 1, 2, 3, \ldots, n$, then

$a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$

(xiii) If $0 < a < 1$ and n is a positive rational number, then

(a) $0 < a^n < 1$ (b) $a^{-n} > 1$

Important Inequality

1. **Arithmetico-Geometric and Harmonic Mean Inequality**

(i) If a, b > 0 and a ≠ b, then

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2}{(1/a)+(1/b)}$$

(ii) if $a_i > 0$, where i = 1,2,3,…,n, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n}$$

$$\ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

(iii) If $a_1, a_2, \ldots, a_n$ are n positive real numbers and $m_1, m_2, \ldots, m_n$ are n positive rational numbers, then

$$\frac{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n}{m_1 + m_2 + \ldots + m_n} > (a_1^{m_1} \cdot a_2^{m_2} \cdot \ldots \cdot a_n^{m_n})^{\frac{1}{m_1 + m_2 + \ldots + m_n}}$$

i.e., Weighted AM > Weighted GM

(iv) If $a_1, a_2, \ldots, a_n$ are n positive distinct real numbers, then

(a)
$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m \text{ if } m < 0 \text{ or } m > 1$$

(b)
$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m, \text{ if } 0 < m < 1$$

(c) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are rational numbers and M is a rational number, then

$$\frac{b_1 a_1^m + b_2 a_2^m + \ldots + b_n a_n^m}{b_1 + b_2 + \ldots + b_n} > \left(\frac{b_1 a_1 + b_2 a_2 + \ldots + b_n a_n}{b_1 + b_2 + \ldots + b_n}\right)^m, \text{if } 0 < m < 1$$

(d)
$$\frac{b_1 a_1^m + b_2 a_2^m + \ldots + b_n a_n^m}{b_1 + b_2 + \ldots + b_n} < \left(\frac{b_1 a_1 + b_2 a_2 + \ldots + b_n a_n}{b_1 + b_2 + \ldots + b_n}\right)^m,$$

$\text{if } 0 < m < 1$

(v) If $a_1, a_2, a_3, \ldots, a_n$ are distinct positive real numbers and p, ,q, r are natural numbers, then

$$\frac{a_1^{p+q+r} + a_2^{p+q+r} + \ldots + a_n^{p+q+r}}{n} > \left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)\left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)\left(\frac{a_1^r + a_2^r + \ldots + a_n^r}{n}\right)$$

2. **Cauchy – Schwartz's inequality**

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots, a_n^2) * (b_1^2 + b_2^2 + \ldots, b_n^2)$

Equality holds, iff $a_1 / b_1 = a_2 / b_2 = a_n / b_n$

3. **Tchebychef's Inequality**

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

(i) If $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

(ii) If If $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

4. **Weierstrass Inequality**

(i) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then for $n \geq 2$

$(1 + a_1)(1 + a_2) \ldots (1 + a_n) > 1 + a_1 + a_2 + \ldots + a_n$

(ii) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then

$(1 - a_1)(1 - a_2) \ldots (1 - a_n) > 1 - a_1 - a_2 - \ldots - a_n$

5. **Logarithm Inequality**

(i) (a) When $y > 1$ and $\log_y x > z \Rightarrow x > y^z$

(b) When $y > 1$ and $\log_y x < z \Rightarrow 0 < x < y^z$

(ii) (a) When $0 < y < 1$ and $\log_y x > z \Rightarrow 0 < x < y^z$

(b) hen $0 < y < 1$ and $\log_y x < z \Rightarrow x > y^z$

Application of Inequalities to Find the Greatest and Least Values

(i) If $x_1, x_2, \ldots, x_n$ are n positive variables such that $x_1 + x_2 + \ldots + x_n = c$ (constant), then the product $x_1 * x_2 * \ldots * x_n$ is greatest when $x_1 = x_2 = \ldots = x_n = c/n$ and the greatest value is $(c/n)^n$.

(ii) If $x_1, x_2, \ldots, x_n$ are positive variables such that $x_1, x_2, \ldots, x_n = c$ (constant), then the sum $x_1 + x_2 + \ldots + x_n$ is least when $x_1 = x_2 = \ldots = x_n = c^{1/n}$ and the least value of the sum is $n(c^{1/n})$.

(iii) If $x_1, x_2, \ldots, x_n$ are variables and $m_1, m_2, \ldots, m_n$ are positive real number such that $x_1 + x_2 + \ldots + x_n = c$ (constant), then $x_1^{m_1} * x_2^{m_2} * \ldots * x_n^{m_n}$ is greatest, when

$x_1 / m_1 = x_2 / m_2 = \ldots = x_n / m_n$

$= x_1 + x_2 + \ldots + x_n / m_1 + m_2 + \ldots + m_n$