Maths Class 11 Chapter 5 Part -1 Quadratic equations

1. **Real Polynomial:** Let $a_0, a_1, a_2, \dots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is called a real polynomial of real variable x with real coefficients.

2. **Complex Polynomial:** If $a_0, a_1, a_2, \dots, a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.

3. **Degree of a Polynomial:** A polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$, real or complex is a polynomial of degree n , if $a_n \neq 0$.

4. **Polynomial Equation:** If f(x) is a polynomial, real or complex, then f(x) = 0 is called a polynomial equation. If f(x) is a polynomial of second degree, then f(x) = 0 is called a quadratic equation .

Quadratic Equation: A polynomial of second degree is called a quadratic polynomial. Polynomials of degree three and four are known as cubic and biquadratic polynomials respectively. A quadratic polynomial f(x) when equated to zero is called quadratic equation. i.e., $ax^2 + bx + c = 0$ where $a \neq 0$.

Roots of a Quadratic Equation: The values of variable x .which satisfy the quadratic equation is called roots of quadratic equation.

Important Points to be Remembered

- An equation of degree n has n roots, real or imaginary .
- Surd and imaginary roots always occur in pairs of a polynomial equation with real coefficients i.e., if $(\sqrt{2} + \sqrt{3}i)$ is a root of an equation, then' $(\sqrt{2} - \sqrt{3}i)$ is also its root. .
- An odd degree equation has at least one real root whose sign is opposite to that of its last' term (constant term), provided that the coefficient of highest degree term is positive.
- Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive has at least two real roots, one positive and one negative.
- If an equation has only one change of sign it has one positive root.
- If all the terms of an equation are positive and the equation involves odd powers of x, then all its roots are complex.

Solution of Quadratic Equation

1.**Factorization Method:** Let $ax^2 + bx + c = \alpha(x - \alpha)(x - \beta) = O$. Then, $x = \alpha$ and $x = \beta$ will satisfy the given equation.

2. **Direct Formula:** Quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha = \frac{-b + \sqrt{D}}{2a}, \beta = \frac{-b - \sqrt{D}}{2a}$$

where D = Δ = b^2 – 4ac is called discriminant of the equation .

Above formulas also known as Sridharacharya formula.

Nature of Roots

Let quadratic equation be $ax^2 + bx + c = 0$, whose discriminant is D.

(i) For $ax^2 + bx + c = 0$; a, b , C ∈ R and a ≠ 0, if

(a) D < => Complex roots

(b) D > 0 => Real and distinct roots

(c) D = 0 => Real and equal roots as α = β = – b/2a

(ii) If a, b, C ∈ Q, a ≠ 0, then

(a) If D > 0 and D is a perfect square => Roots are unequal and rational.

(b) If D > 0, a = 1; b, c ∈ I and D is a perfect square. => Roots are integral. .

(c) If D > and D is not a perfect square. => Roots are irrational and unequal.

(iii) **Conjugate Roots** The irrational and complex roots of a quadratic equation always occur in pairs. Therefore,

(a) If one root be α + iβ, then other root will be α – iβ.

(b) If one root be α + √β, then other root will be α – √β.

(iv) If D, and D2 be the discriminants of two quadratic equations, then

(a) If $D_1 + D_2 \geq 0$, then At least one of D_1 and $D_2 \geq 0$ If $D_1 < 0$, then $D_2 > 0$,

(b) If $D_1 + D_2 < 0$, then At least one of D_1 and $D_2 < 0$ If $D_1 > 0$, then $D_2 < 0$

Roots Under Particular Conditions

For the quadratic equation $ax^2 + bx + e = 0$.

(i) If $b = 0$ => Roots are real/complex as ($c < 0/c > 0$) and equal in magnitude but of opposite sign.

(ii) If $c = 0$ => One roots is zero, other is $– b / a$.

(iii) If $b = C = 0$ => Both roots are zero.

(iv) If $a = c$ => Roots are reciprocal to each other.

(v) If $a > 0, c < 0, a < 0, c > 0\}$ => Roots are of opposite sign.

(vi) If $a > 0, b > 0, c > 0, a < 0, b < 0, c < 0\}$ => Both roots are negative, provided $D \geq 0$

(vii) If $a > 0, b < 0, c > 0, a < 0, b > 0, c < 0\}$ => Both roots are positive, provided $D \geq 0$

(viii) If sign of a = sign of b ≠ sign of c => Greater root in magnitude is negative.

(ix) If sign of b = sign of c ≠. sign of a => Greater root in magnitude is positive.

(x) If $a + b + c = 0$ => One root is 1 and second root is c/a.

Relation between Roots and Coefficients

1. **Quadratic Equation:** If roots of quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are α and β, then Sum of roost = $S = \alpha + \beta = -b/a$ = – coefficient of x / coefficient of x^2 Product of roots = $P = \alpha * \beta = c/a$ = constant term / coefficient of x^2

2. **Cubic Equation:** If α, β and γ are the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$.

Then,

$$\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

3. **Biquadratic Equation:** If α, β, γ and δ are the roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma = -\frac{b}{a},$$

$$S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2\frac{c}{a} = \frac{c}{a}$$

$$S_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

$$S_3 = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a}$$

$$S_4 = \alpha \cdot \beta \cdot \gamma \cdot \delta = (-1)^4 \frac{e}{a} = \frac{e}{a}$$

Symmetric Roots: If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are α and β, then

(i) $(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm \frac{\sqrt{b^2 - 4ac}}{a} = \frac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm \frac{b\sqrt{b^2 - 4ac}}{a^2} = \frac{\pm b\sqrt{D}}{a^2}$

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = -\frac{b(b^2 - 3ac)}{a^3}$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta) = \frac{\pm(b^2 - ac)\sqrt{b^2 - 4ac}}{a^3}$

(vi) $\alpha^4 + \beta^4 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2 = \left(\frac{b^2 - 2ac}{a^2}\right)^2 - \frac{2c^2}{a^2}$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) = \frac{\pm b(b^2 - 2ac)\sqrt{b^2 - 4ac}}{a^4}$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta = \frac{b^2 - ac}{a^2}$

(ix) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{b^2 - 2ac}{ac}$

(x) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = -\frac{bc}{a^2}$

(xi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2} = \frac{b^2 D + 2a^2c^2}{a^2c^2}$

Formation of Polynomial Equation from Given Roots

If a_1, a_2 a_3,…, an are the roots of an nth degree equation, then the equation is $x^n - S_1X^{n-1} + S_2X^{n-2} - S_3X^{n-3} + \ldots + (\_l)^n S_n = 0$ where S_n denotes the sum of the products of roots taken n at a time.

1. **Quadratic Equation**

If α and β are the roots of ‘a quadratic equation, then the equation is $x^2 - S_1X + S_2 = 0$

i.e., $x^2 - (\alpha + \beta) x + \alpha\beta = 0$

2. **Cubic Equation**

If α, β and γ are the roots of cubic equation, then the equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$
$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$$

3. **Biquadratic Equation**

If α, β, γ and δ are the roots of a biquadratic equation, then the equation is

$$x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0$$
$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha)x + \alpha\beta\gamma\delta = 0$$

Equation In Terms of the Roots of another Equation

If α, β are roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are.

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (replace x by $-x$)

(ii) $\alpha^n, \beta^n; n \in N \Rightarrow a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (replace x by $x^{1/n}$)

(iii) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$ (replace x by x/k)

(iv) $k + \alpha, k + \beta \Rightarrow a(x - k)^2 + b(x - k) + c = 0$ (replace x by $(x - k)$)

(v) $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (replace x by kx)

(vi) $\alpha^{1/n}, \beta^{1/n}; n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$ (replace x by x^n)

The quadratic function $f(x) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is always resolvable into linear factor, iff

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Condition for Common Roots in a Quadratic Equation

1. **Only One Root is Common**

If α be the common root of quadratic equations

$a_1x^2 + b_1x + C_1 = 0$,

and $a_2x^2 + b_2x + C_2 = 0$,

then $a_1a^2 + b_1\alpha + C_1 = 0$,

and $a_2a^2 + b_2\alpha + C_2 = 0$,

By Cramer's Rule

$$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}, \alpha \neq 0$$

Hence, the condition for only one root common is

$(c_1a_2 – c_2a_1)_2 = (b_1c_2 – b_2c_1)(a_1b_2 – a_2b_1)$

2. **Both Roots are Common**

The required condition is

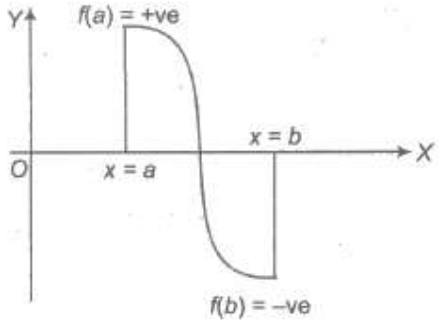
$a_1 / a_2 = b_1 / b_2 = c_1 / c_2$

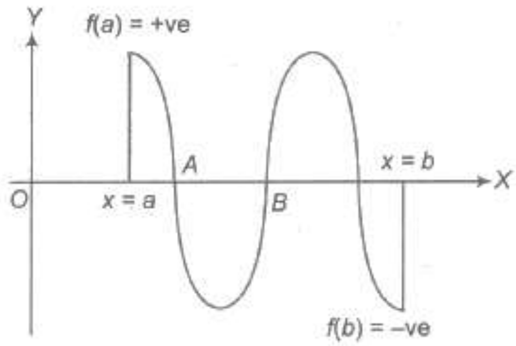
(i) To find the common root of two equations, make the coefficient of second degree term in the two equations equal and subtract. The value of x obtained is the required common root.

(ii) Two different quadratic equations with rational coefficient can not have single common root which is complex or irrational as imaginary and surd roots always occur in pair.

Properties of Quadratic Equation

(i) f(a) . f(b) < 0, then at least one or in general odd number of roots of the equation f(x) = 0 lies between a and b.





(ii) f(a) . f(b) > 0, then in general even number of roots of the equation f(x) = 0 lies between a and b or no root exist f(a) = f(b), then there exists a point c between a and b such that f'(c) = 0, a < c < b.

(iii) If the roots of the quadratic equation $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ are in the ratio (i.e., &alpha$_1$;/β_1 = &alpha$_2$;/β_2), then

$b_1^2 / b_2^2 = a_1c_1 / a_2c_2$.

(iv) If one root is k times the other root of the quadratic equation $ax^2 + bx + c = 0$,then

$(k + 1)^2 / k = b^2 / ac$

Quadratic Expression

An expression of the form $ax^2 + bx + c$, where a, b, c ∈ R and a ≠ 0 is called a quadratic expression in x .

1. **Graph of a Quadratic Expression**

We have

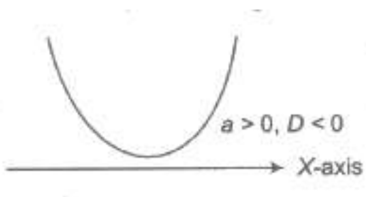
$y = ax^2 + bx + c = f(x)$

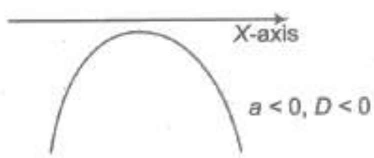
$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \Rightarrow y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Let y + D/4a = Y and x + D / 2a = X

Y = a * X^2 => X^2 = Y / a

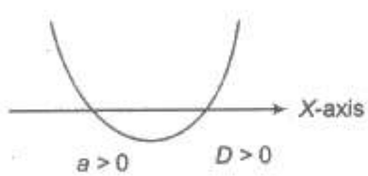
(i) The graph of the curve y = f(x) is parabolic.
(ii) The axis of parabola is X = 0 or x + b / 2a = 0 i.e., (parallel to Y-axis). •
(iii) If a > 0, then the parabola opens upward.
If a < 0, then the parabola opens downward.

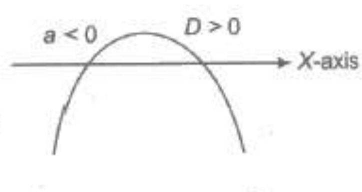




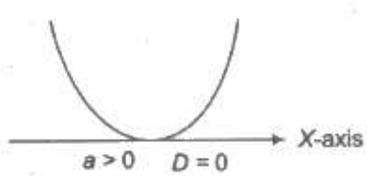
2. **Position of** y = ax^2 +bx + c **with Respect to Axes.**

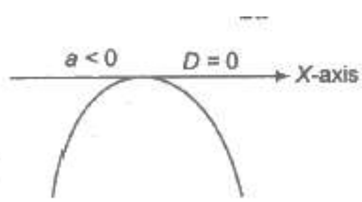
(i) For D > 0, parabola cuts X-axis in two real and distinct points
i.e, x = -b ± √D / 2a





(ii) For D = 0, parabola touch X-axis in one point, x = – b/2a.





(iii) For D < O,parabola does not cut X-axis (i.e., imaginary value of x).