16. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

18. In triangle ABC (Fig 10.18), which of the following is not true:

(A) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(B) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(C) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(D) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

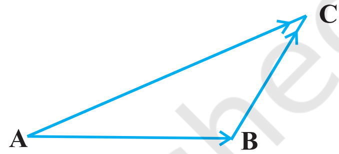

Fig 10.18

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(A) $\vec{b} = \lambda\vec{a}$, for some scalar λ

(B) $\vec{a} = \pm\vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are not proportional

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

10.6 Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiply them in two ways, namely, multiplication of two functions pointwise and composition of two functions. Similarly, multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector. Based upon these two types of products for vectors, they have found various applications in geometry, mechanics and engineering. In this section, we will discuss these two types of products.

10.6.1 *Scalar (or dot) product of two vectors*

Definition 2 The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$

where, θ is the angle between $\vec{a}$ and $\vec{b}, 0 \le \theta \le \pi$ (Fig 10.19).

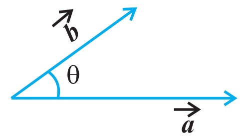

Fig 10.19

If either $\vec{a} = 0$ or $\vec{b} = 0$ then θ is not defined, and in this case, we define $\vec{a} \cdot \vec{b} = 0$

Observations

1. $\vec{a} \cdot \vec{b}$ is a real number.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors, then $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e.
   $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
3. If θ = 0, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$
   In particular, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, as θ in this case is 0.
4. If θ = π, then $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = -|\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|$
   In particular, $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$, as θ in this case is π.
5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, we have
   $$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1,$$
   $$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{i}} = 0$$
6. The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is given by
   $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}, \text{ or } \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$
7. The scalar product is commutative. i.e.
   $$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \text{(Why?)}$$

Two important properties of scalar product

Property 1 (Distributivity of scalar product over addition) Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Property 2 Let $\vec{a}$ and $\vec{b}$ be any two vectors, and l be any scalar. Then

$$(\lambda\vec{a})\cdot\vec{b} = (\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$$

If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then their scalar product is given as

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\
&= a_1\hat{i}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_2\hat{j}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_3\hat{k}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\
&= a_1b_1(\hat{i}\cdot\hat{i})+a_1b_2(\hat{i}\cdot\hat{j})+a_1b_3(\hat{i}\cdot\hat{k})+a_2b_1(\hat{j}\cdot\hat{i})+a_2b_2(\hat{j}\cdot\hat{j})+a_2b_3(\hat{j}\cdot\hat{k})\\
&\quad + a_3b_1(\hat{k}\cdot\hat{i})+a_3b_2(\hat{k}\cdot\hat{j})+a_3b_3(\hat{k}\cdot\hat{k}) \quad \text{(Using the above Properties 1 and 2)}\\
&= a_1b_1 + a_2b_2 + a_3b_3 \quad \text{(Using Observation 5)}
\end{aligned}$$

Thus $\quad \vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

10.6.2 *Projection of a vector on a line*

Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line ***l*** (say), in the ***anticlockwise direction*** (Fig 10.20). Then the projection of $\overrightarrow{AB}$ on ***l*** is a vector $\vec{p}$ (say) with magnitude $|\overrightarrow{AB}|\,|\cos\theta|$, and the direction of $\vec{p}$ being the same (or opposite) to that of the line ***l***, depending upon whether cos θ is positive or negative. The vector $\vec{p}$

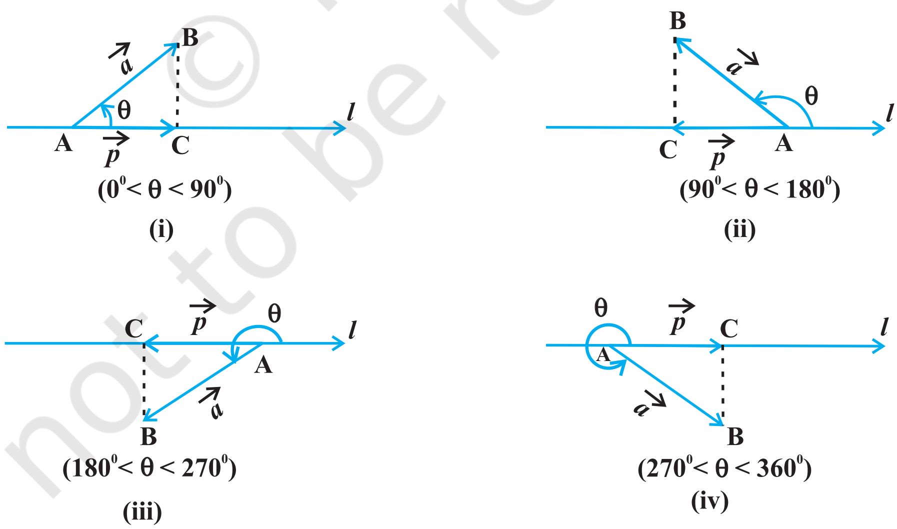

Fig 10.20

is called the ***projection vector***, and its magnitude $|\vec{p}|$ is simply called as the ***projection*** of the vector $\overrightarrow{AB}$ on the directed line $\boldsymbol{l}$.

For example, in each of the following figures (Fig 10.20 (i) to (iv)), projection vector of $\overrightarrow{AB}$ along the line $\boldsymbol{l}$ is vector $\overrightarrow{AC}$.

Observations

1. If $\hat{\boldsymbol{p}}$ is the unit vector along a line $\boldsymbol{l}$, then the projection of a vector $\vec{a}$ on the line $\boldsymbol{l}$ is given by $\vec{a} \cdot \hat{\boldsymbol{p}}$.
2. Projection of a vector $\vec{a}$ on other vector $\vec{\boldsymbol{b}}$, is given by
$$\vec{a} \cdot \hat{\boldsymbol{b}}, \quad \text{or} \quad \vec{a} \cdot \left(\frac{\vec{b}}{|\vec{b}|}\right), \text{ or } \frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b})$$
3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.
4. If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

Remark If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as

$$\cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Also, note that $|\vec{a}|\cos\alpha$, $|\vec{a}|\cos\beta$ and $|\vec{a}|\cos\gamma$ are respectively the projections of $\vec{a}$ along OX, OY and OZ. i.e., the scalar components $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ of the vector $\vec{a}$, are precisely the projections of $\vec{a}$ along ***x***-axis, ***y***-axis and ***z***-axis, respectively. Further, if $\vec{a}$ is a unit vector, then it may be expressed in terms of its direction cosines as

$$\vec{a} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$

Example 13 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $\vec{a} \cdot \vec{b} = 1$.

Solution Given $\vec{a} \cdot \vec{b} = 1, |\vec{a}| = 1$ and $|\vec{b}| = 2$. We have

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Example 14 Find angle 'θ' between the vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$.

Solution The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Now $\quad \vec{a}\cdot\vec{b} = (\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}})\cdot(\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}) = 1-1-1 = -1.$

Therefore, we have $\quad \cos\theta = \frac{-1}{3}$

hence the required angle is $\quad \theta = \cos^{-1}\left(-\frac{1}{3}\right)$

Example 15 If $\vec{a}=5\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b}=\hat{i}+3\hat{j}-5\hat{k}$, then show that the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular.

Solution We know that two nonzero vectors are perpendicular if their scalar product is zero.

Here $\quad \vec{a}+\vec{b} = (5\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}})+(\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}) = 6\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-8\hat{\boldsymbol{k}}$

and $\quad \vec{a}-\vec{b} = (5\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}})-(\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}) = 4\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$

So $\quad (\vec{a}+\vec{b})\,.\,(\vec{a}-\vec{b}) = (6\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-8\hat{\boldsymbol{k}})\cdot(4\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}) = 24-8-16 = 0.$

Hence $\quad \vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors.

Example 16 Find the projection of the vector $\vec{a}=2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ on the vector $\vec{b}=\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$.

Solution The projection of vector $\vec{\boldsymbol{a}}$ on the vector $\vec{\boldsymbol{b}}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{(2\times1+3\times2+2\times1)}{\sqrt{(1)^2+(2)^2+(1)^2}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}$$

Example 17 Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$.

Solution We have

$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}.\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

$$= (2)^2 - 2(4) + (3)^2$$

Therefore $\qquad |\vec{a}-\vec{b}| = \sqrt{5}$

Example 18 If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$, then find $|\vec{x}|$.

Solution Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$. Also,

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$$

or $\qquad \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{a} = 8$

or $\qquad |\vec{x}|^2 - 1 = 8$ i.e. $|\vec{x}|^2 = 9$

Therefore $\qquad |\vec{x}| = 3$ (as magnitude of a vector is non negative).

Example 19 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$ (Cauchy-Schwartz inequality).

Solution The inequality holds trivially when either $\vec{\boldsymbol{a}} = \vec{0}$ or $\vec{\boldsymbol{b}} = \vec{0}$. Actually, in such a situation we have $|\vec{a}\cdot\vec{b}| = 0 = |\vec{a}||\vec{b}|$. So, let us assume that $|\vec{a}| \ne 0 \ne |\vec{b}|$. Then, we have

$$\frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \le 1$$

Therefore $\qquad |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$

Example 20 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (triangle inequality).

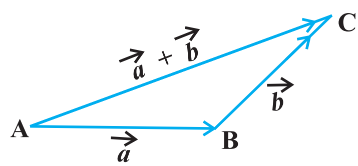

Fig 10.21

Solution The inequality holds trivially in case either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ (How?). So, let $|\vec{a}| \ne \vec{0} \ne |\vec{b}|$. Then,

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \qquad \text{(scalar product is commutative)}$$

$$\le |\vec{a}|^2 + 2|\vec{a}\cdot\vec{b}| + |\vec{b}|^2 \qquad \text{(since } \boldsymbol{x} \le |\boldsymbol{x}|\ \forall \boldsymbol{x} \in \mathbf{R}\text{)}$$

$$\le |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \qquad \text{(from Example 19)}$$

$$= (|\vec{a}| + |\vec{b}|)^2$$

Hence $$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

Remark If the equality holds in triangle inequality (in the above Example 20), i.e.

$$|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|,$$

then $$|\overrightarrow{AC}| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

showing that the points A, B and C are collinear.

Example 21 Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$ are collinear.

Solution We have

$$\overrightarrow{AB} = (1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k} = 3\hat{i}-\hat{j}-2\hat{k},$$

$$\overrightarrow{BC} = (7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k} = 6\hat{i}-2\hat{j}-4\hat{k},$$

$$\overrightarrow{AC} = (7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k} = 9\hat{i}-3\hat{j}-6\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{14},\ |\overrightarrow{BC}| = 2\sqrt{14} \text{ and } |\overrightarrow{AC}| = 3\sqrt{14}$$

Therefore $$|\overrightarrow{AC}| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

Hence the points A, B and C are collinear.

Note In Example 21, one may note that although $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ but the points A, B and C do not form the vertices of a triangle.

EXERCISE 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.
2. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$
3. Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.
4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
5. Show that each of the given three vectors is a unit vector:

   $$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\ \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\ \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$

   Also, show that they are mutually perpendicular to each other.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

7. Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

12. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

13. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.

17. Show that the vectors $2\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}$ form the vertices of a right angled triangle.

18. If $\vec{a}$ is a nonzero vector of magnitude '***a***' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $\boldsymbol{a} = |\lambda|$ (D) $\boldsymbol{a} = 1/|\lambda|$

10.6.3 *Vector (or cross) product of two vectors*

In Section 10.2, we have discussed on the three dimensional right handed rectangular coordinate system. In this system, when the positive ***x***-axis is rotated counterclockwise