3. Find a unit vector in the direction of PQ, where P and Q have co-ordinates (5,0, 8) and (3, 3,2), respectively.

Sol. We have points $P(5, 0, 8)$ and $Q(3, 3, 2)$, respectively.

or $\overrightarrow{OP} = 5\hat{i} + 0\hat{j} + 8\hat{k}$

and $\overrightarrow{OQ} = 3\hat{i} + 3\hat{j} + 2\hat{k}$

$\therefore$
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (3\hat{i} + 3\hat{j} + 2\hat{k}) - (5\hat{i} + 0\hat{j} + 8\hat{k})$$
$$= -2\hat{i} + 3\hat{j} - 6\hat{k}$$

$\therefore$ Unit vector in the direction of $\overrightarrow{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}}$$
$$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{49}}$$
$$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7}$$

4. If $\vec{a}$ and b are the position vectors of A and B, respectively, find the position vector of a point C on BA produced such that $BC = 1.5\ BA$.

Sol. We have $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Let $\overrightarrow{OC} = \vec{c}$

$$BC = 1.5\ BA$$

$\Rightarrow$ $\overrightarrow{BC} = 1.5\overrightarrow{BA}$ (as points A, B, C are collinear).

$\Rightarrow$ $\overrightarrow{OC} - \overrightarrow{OB} = 1.5(\overrightarrow{OA} - \overrightarrow{OB})$

$\Rightarrow$ $2(\vec{c} - \vec{b}) = 3(\vec{a} - \vec{b})$

$\Rightarrow$ $2\vec{c} - 2\vec{b} = 3\vec{a} - 3\vec{b}$

$\Rightarrow$ $2\vec{c} = 3\vec{a} - 3\vec{b} + 2\vec{b}$

$\Rightarrow$ $\vec{c} = \dfrac{3\vec{a} - \vec{b}}{2}$

5. Using vectors, find the value of k such that the points (k, -10, 3), (1, -1, 3) and (3, 5, 3) are collinear.

Sol. Let the points are $A(k, -10, 3)$, $B(1, -1, 3)$ and $C(3, 5, 4)$

$\therefore$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (\hat{i} - \hat{j} + 3\hat{k}) - (k\hat{i} - 10\hat{j} + 3\hat{k}) = (1-k)\hat{i} + 9\hat{j}$$

and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$$= (3\hat{i} + 5\hat{j} + 3\hat{k}) - (\hat{i} - \hat{j} + 3\hat{k}) = 2\hat{i} + 6\hat{j}$$

If point A, B, C are collinear then there exists some scalar 'λ' such that

$\Rightarrow$ $\overrightarrow{AB} = \lambda\overrightarrow{BC}$

$$(1-k)\hat{i} + 9\hat{j} = \lambda(2\hat{i} + 6\hat{j})$$

Comparing coefficients we get $1 - k = 2\lambda$ and $9 = 6\lambda$

$\therefore$ $3 - 3k = 9$

$\therefore$ $k = -2$

12. If A, B, C, D are the points with position vectors $\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, respectively, find the projection of $\overrightarrow{AB}$ along $\overrightarrow{CD}$.

Sol. We have $\overrightarrow{OA}=\hat{i}+\hat{j}-\hat{k}, \overrightarrow{OB}=2\hat{i}-\hat{j}+3\hat{k},$,

$$\overrightarrow{OC}=2\hat{i}-3\hat{k} \text{ and } \overrightarrow{OD}=3\hat{i}-2\hat{j}+\hat{k}$$

$$\therefore \quad \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

$$=(2\hat{i}-\hat{j}+3\hat{k})-(\hat{i}+\hat{j}-\hat{k})=\hat{i}-2\hat{j}+4\hat{k}$$

and $$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}$$

$$=(3\hat{i}-2\hat{j}+\hat{k})-(2\hat{i}-3\hat{k})=\hat{i}-2\hat{j}+4\hat{k}$$

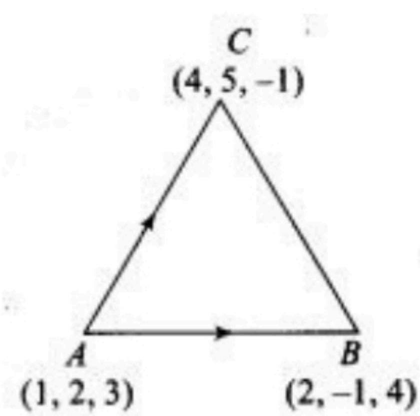
Now the projection of $\overrightarrow{AB}$ along $\overrightarrow{CD}$

$$=\overrightarrow{AB}\cdot\frac{\overrightarrow{CD}}{|\overrightarrow{CD}|}$$

$$=\frac{(\hat{i}-2\hat{j}+4\hat{k})\cdot(\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{1^2+2^2+4^2}}$$

$$=\frac{1+4+16}{\sqrt{21}}=\frac{21}{\sqrt{21}}=\sqrt{21} \text{ units}$$

13. Using vectors, find the area of the triangle ABC with vertices A(l, 2, 3), 5(2, -1, 4) and C(4,5,-l).



Sol. Here, $\overrightarrow{AB}=(2-1)\hat{i}+(-1-2)\hat{j}+(4-3)\hat{k}$

$$=\hat{i}-3\hat{j}+\hat{k}$$

And $$\overrightarrow{AC}=(4-1)\hat{i}+(5-2)\hat{j}+(-1-3)\hat{k}$$

$$=3\hat{i}+3\hat{j}-4\hat{k}$$

$$\therefore \quad \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}$$

$$=\hat{i}(12-3)-\hat{j}(-4-3)+\hat{k}(3+9)=9\hat{i}+7\hat{j}+12\hat{k}$$

$$\therefore \text{ Area of } \Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{9^2+7^2+12^2}=\frac{1}{2}\sqrt{274} \text{ sq. units}$$

10. If $\vec{a}+\vec{b}+\vec{c}=0$, then show that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. Interpret the result geometrically.

Sol. Since $\quad \vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \quad \vec{b}=-\vec{c}-\vec{a}$

Now, $\quad \vec{a}\times\vec{b}=\vec{a}\times(-\vec{c}-\vec{a})$

$$=-\vec{a}\times\vec{c}-\vec{a}\times\vec{a}=-\vec{a}\times\vec{c}$$

$\Rightarrow \quad \vec{a}\times\vec{b}=\vec{c}\times\vec{a}$ (i)

Also, $\quad \vec{b}\times\vec{c}=(-\vec{c}-\vec{a})\times\vec{c}$

$$=-\vec{c}\times\vec{c}-\vec{a}\times\vec{c}=-\vec{a}\times\vec{c}$$

$\Rightarrow \quad \vec{b}\times\vec{c}=\vec{c}\times\vec{a}$ (ii)

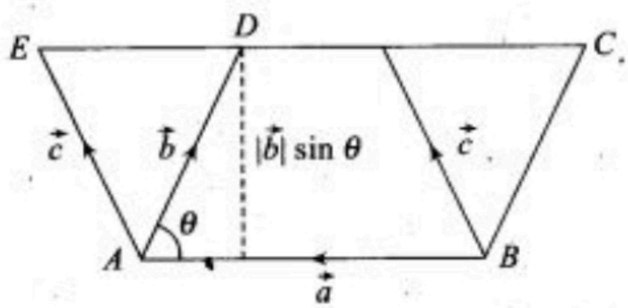
From (i) and (ii), $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Geometrical interpretation of the result

If $ABCD$ is a parallelogram such that $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{AD}=\vec{b}$ and these adjacent sides are making angle θ between each other, then

Area of parallelogram $ABCD=|\vec{a}||\vec{b}|\sin\theta=|\vec{a}\times\vec{b}|$

Since, parallelogram on the same base and between the same parallels are equal



we have, $|\vec{a}\times\vec{b}|=|\vec{a}\times\vec{c}|=|\vec{b}\times\vec{c}|$

This also implies that, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}=\vec{b}\times\vec{c}$

So, area of the parallelograms formed by taking any two sides represented by $\vec{a},\vec{b}$ and $\vec{c}$ as adjacent are equal.

11. Find the sine of the angle between the vectors $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$.

Sol. We know that, angle between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$=\frac{(3\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{3^2+1^2+2^2}\sqrt{2^2+(-2)^2+4^2}}$$

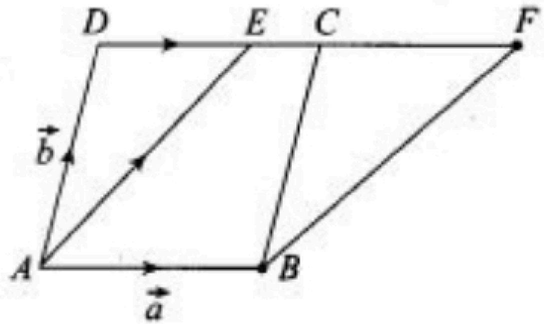
$\therefore \quad \cos\theta=\frac{3}{\sqrt{21}}$

$\therefore \quad \sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{9}{21}}=\frac{2}{\sqrt{7}}$

14. Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.

Sol. Let *ABCD* and *ABFE* are parallelograms on the same base *AB* and between the same parallel lines *AB* and *DF*.

Here, $AB \parallel CD$ and $AE \parallel BF$



Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

$\therefore$ Area of parallelogram *ABCD*

$= \vec{a} \times \vec{b}$

Now, area of parallelogram *ABEF*

$= \overrightarrow{AB} \times \overrightarrow{AE}$

$= \overrightarrow{AB} \times (\overrightarrow{AD} + \overrightarrow{DE})$

$= \overrightarrow{AB} \times (\vec{b} + k\vec{a})$

$= \vec{a} \times (\vec{b} + k\vec{a})$

$= (\vec{a} \times \vec{b}) + (\vec{a} \times k\vec{a})$

$= (\vec{a} \times \vec{b}) + k(\vec{a} \times \vec{a})$

$= (\vec{a} \times \vec{b})$ $\quad [\because \vec{a} \times \vec{a} = 0]$

= Area of parallelogram *ABCD*

Hence proved.