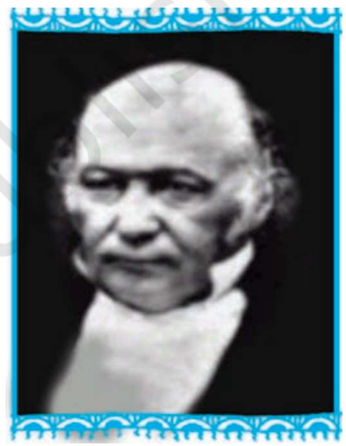
Chapter 10

VECTOR ALGEBRA

❖ ***In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure. – HERMAN HANKEL*** ❖

10.1 Introduction



W.R. Hamilton (1805-1865)

In our day to day life, we come across many queries such as – What is your height? How should a football player hit the ball to give a pass to another player of his team? Observe that a possible answer to the first query may be 1.6 meters, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors.* In mathematics, physics and engineering, we frequently come across with both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two type of properties, when considered together give a full realisation to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

10.2 Some Basic Concepts

Let '*l*' be any straight line in plane or three dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a *directed line* (Fig 10.1 (i), (ii)).

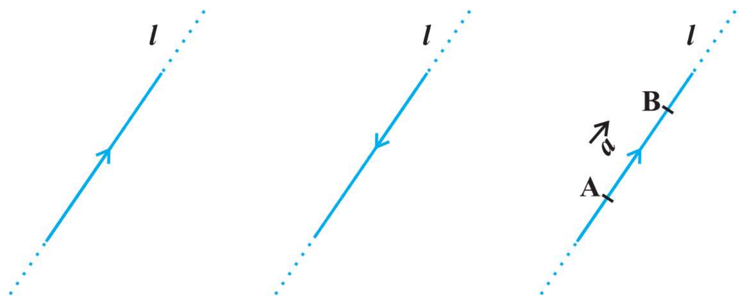


Fig 10.1

Now observe that if we restrict the line *l* to the line segment AB, then a magnitude is prescribed on the line *l* with one of the two directions, so that we obtain a ***directed line segment*** (Fig 10.1(iii)). Thus, a directed line segment has magnitude as well as direction.

Definition 1 A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

The point A from where the vector $\overrightarrow{AB}$ starts is called its ***initial point***, and the point B where it ends is called its ***terminal point***. The distance between initial and terminal points of a vector is called the ***magnitude*** (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or ***a***. The arrow indicates the direction of the vector.

Note Since the length is never negative, the notation $|\vec{a}| < 0$ has no meaning.

Position Vector

From Class XI, recall the three dimensional right handed rectangular coordinate system (Fig 10.2(i)). Consider a point P in space, having coordinates (*x*, *y*, *z*) with respect to the origin O (0, 0, 0). Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the ***position vector*** of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice, the position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$, etc., respectively (Fig 10.2 (ii)).

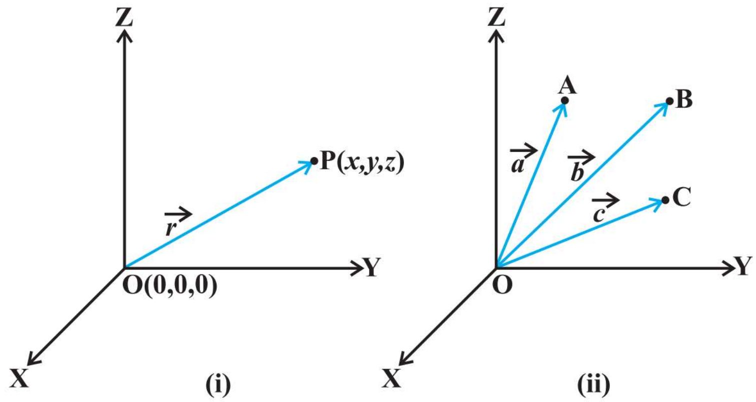


Fig 10.2

Direction Cosines

Consider the position vector $\overrightarrow{OP}\left(\text{or } \vec{r}\right)$ of a point P(*x*, *y*, *z*) as in Fig 10.3. The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of *x*, *y* and *z*-axes respectively, are called its ***direction angles***. The cosine values of these angles, i.e., cos α, cos β and cos γ are called ***direction cosines*** of the vector $\vec{r}$, and usually denoted by ***l***, ***m*** and ***n***, respectively.

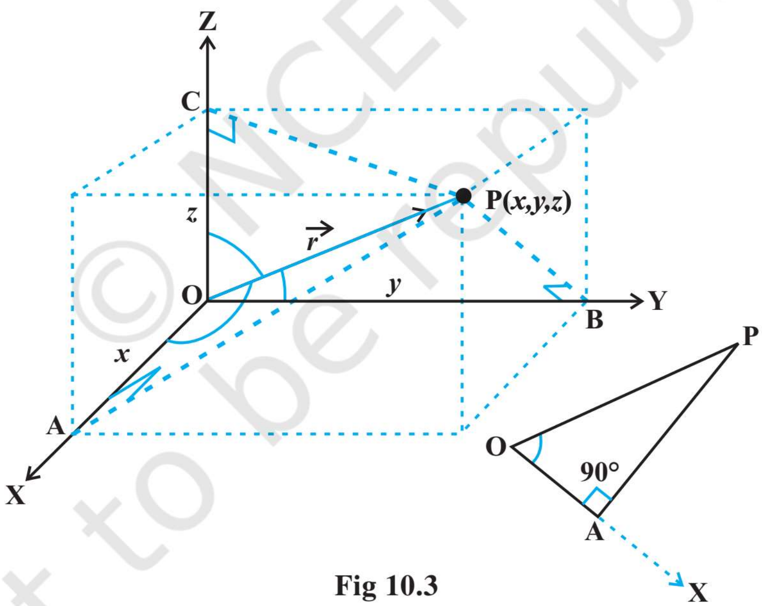


Fig 10.3

From Fig 10.3, one may note that the triangle OAP is right angled, and in it, we have $\cos\alpha = \frac{x}{r}$ $\left(r \text{ stands for } |\vec{r}|\right)$. Similarly, from the right angled triangles OBP and OCP, we may write $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$. Thus, the coordinates of the point P may also be expressed as (*lr*, *mr*,*nr*). The numbers *lr*, *mr* and *nr*, proportional to the direction cosines are called as ***direction ratios*** of vector $\vec{r}$, and denoted as *a*, *b* and *c*, respectively.

Note One may note that $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.

10.3 Types of Vectors

Zero Vector A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}, \overrightarrow{BB}$ represent the zero vector,

Unit Vector A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$.

Coinitial Vectors Two or more vectors having the same initial point are called coinitial vectors.

Collinear Vectors Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Equal Vectors Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

Negative of a Vector A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called *negative* of the given vector. For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Remark The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called *free vectors*. Throughout this chapter, we will be dealing with free vectors only.

Example 1 Represent graphically a displacement of 40 km, 30° west of south.

Solution The vector $\overrightarrow{OP}$ represents the required displacement (Fig 10.4).

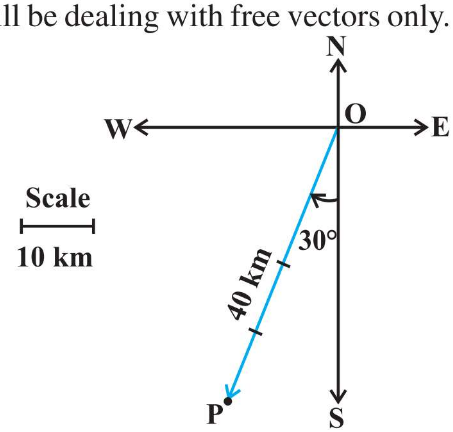


Fig 10.4

Example 2 Classify the following measures as scalars and vectors.

(i) 5 seconds

(ii) 1000 cm^3

(iii) 10 Newton (iv) 30 km/hr (v) 10 g/cm^3
(vi) 20 m/s towards north

Solution

(i) Time-scalar (ii) Volume-scalar (iii) Force-vector
(iv) Speed-scalar (v) Density-scalar (vi) Velocity-vector

Example 3 In Fig 10.5, which of the vectors are:

(i) Collinear (ii) Equal (iii) Coinitial

Solution

(i) Collinear vectors : $\vec{a}$, $\vec{c}$ and $\vec{d}$.

(ii) Equal vectors : $\vec{a}$ and $\vec{c}$.

(iii) Coinitial vectors : $\vec{b}$, $\vec{c}$ and $\vec{d}$.

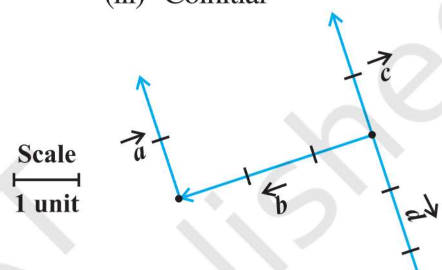


Fig 10.5

EXERCISE 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.
2. Classify the following measures as scalars and vectors.
   (i) 10 kg (ii) 2 meters north-west (iii) 40°
   (iv) 40 watt (v) 10^{-19} coulomb (vi) 20 m/s^2
3. Classify the following as scalar and vector quantities.
   (i) time period (ii) distance
   (iii) force
   (iv) velocity (v) work done
4. In Fig 10.6 (a square), identify the following vectors.
   (i) Coinitial (ii) Equal
   (iii) Collinear but not equal

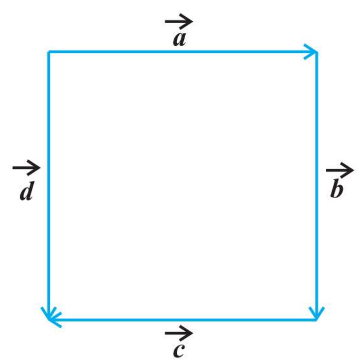


Fig 10.6

5. Answer the following as true or false.
   (i) $\vec{a}$ and $-\vec{a}$ are collinear.
   (ii) Two collinear vectors are always equal in magnitude.
   (iii) Two vectors having same magnitude are collinear.
   (iv) Two collinear vectors having the same magnitude are equal.

10.4 Addition of Vectors

A vector $\overrightarrow{AB}$ simply means the displacement from a point A to the point B. Now consider a situation that a girl moves from A to B and then from B to C (Fig 10.7). The net displacement made by the girl from point A to the point C, is given by the vector $\overrightarrow{AC}$ and expressed as

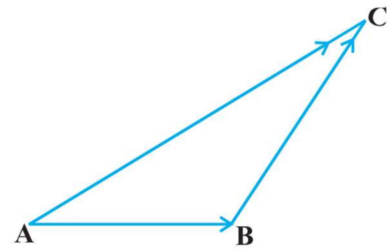


Fig 10.7

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the ***triangle law of vector addition.***

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig 10.8 (i)), then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other (Fig 10.8(ii)).

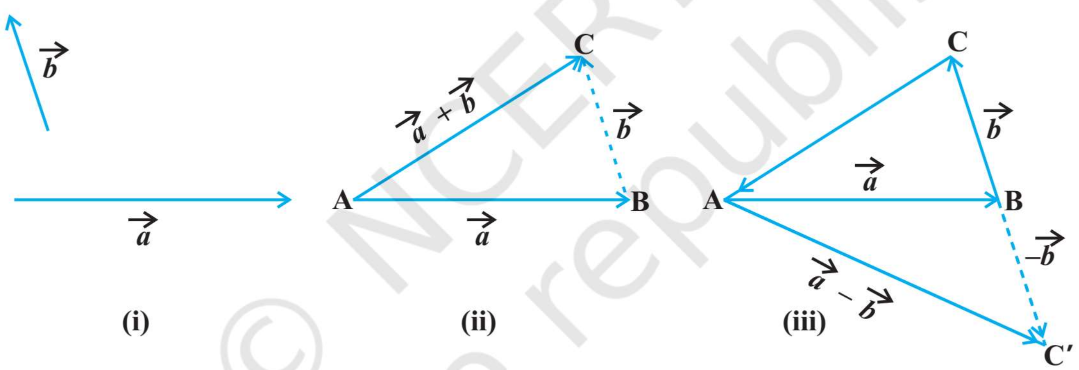


Fig 10.8

For example, in Fig 10.8 (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction, so that it's initial point coincides with the terminal point of $\vec{a}$. Then, the vector $\vec{a} + \vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ i.e., in triangle ABC (Fig 10.8 (ii)), we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Now again, since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig 10.8(iii)).

Now, construct a vector $\overrightarrow{BC'}$ so that its magnitude is same as the vector $\overrightarrow{BC}$, but the direction opposite to that of it (Fig 10.8 (iii)), i.e.,

$$\overrightarrow{BC'} = -\overrightarrow{BC}$$

Then, on applying triangle law from the Fig 10.8 (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

The vector $\overrightarrow{AC'}$ is said to represent the *difference of* $\vec{a}$ and $\vec{b}$.

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors–one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat in actual starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig 10.9), then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the *parallelogram law of vector addition.*

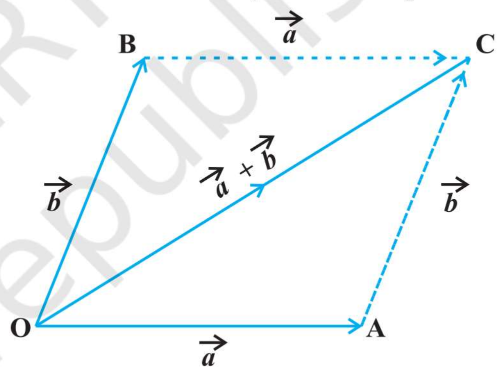


Fig 10.9

Note From Fig 10.9, using the triangle law, one may note that

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

or $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \qquad (\text{since } \overrightarrow{AC} = \overrightarrow{OB})$$

which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

Properties of vector addition

Property 1 For any two vectors $\vec{a}$ and $\vec{b}$,

$$\vec{a}+\vec{b} = \vec{b}+\vec{a} \qquad \text{(Commutative property)}$$

Proof Consider the parallelogram ABCD (Fig 10.10). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, then using the triangle law, from triangle ABC, we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

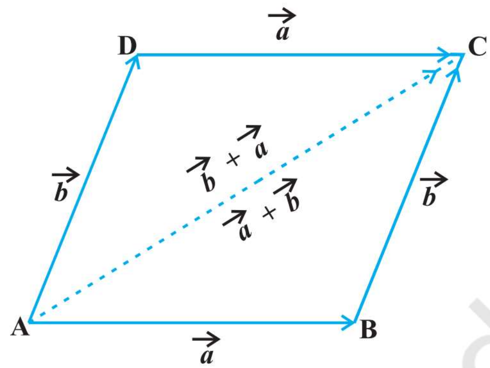


Fig 10.10

Now, since the opposite sides of a parallelogram are equal and parallel, from Fig 10.10, we have, $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$. Again using triangle law, from triangle ADC, we have

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

Hence $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Property 2 For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(Associative property)}$$

Proof Let the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$, respectively, as shown in Fig 10.11(i) and (ii).

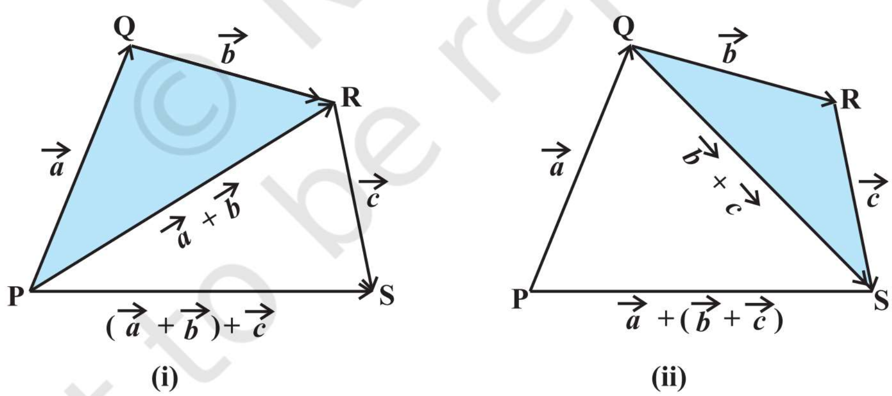


Fig 10.11

Then $\vec{a} + \vec{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

and $\vec{b} + \vec{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$

So $(\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$

and $\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$

Hence $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Remark The associative property of vector addition enables us to write the sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ as $\vec{a} + \vec{b} + \vec{c}$ without using brackets.

Note that for any vector $\vec{a}$, we have

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

Here, the zero vector $\vec{0}$ is called the ***additive identity*** for the vector addition.

10.5 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that, $\lambda\vec{a}$ is also a vector, collinear to the vector $\vec{a}$. The vector $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualisation of multiplication of a vector by a scalar is given in Fig 10.12.

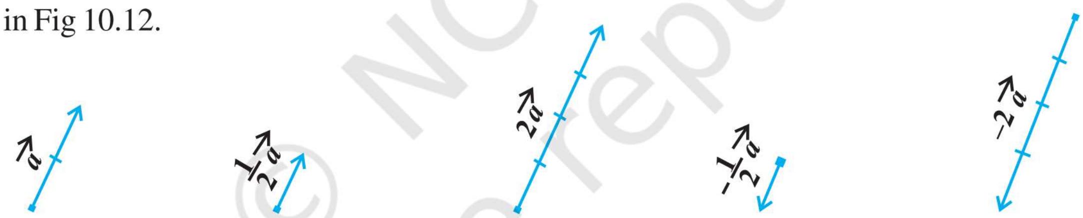


Fig 10.12

When $\lambda = -1$, then $\lambda\vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$. The vector $-\vec{a}$ is called the ***negative*** (or ***additive inverse***) ***of vector*** $\vec{a}$ and we always have

$$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$$

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$ i.e. $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

So, $\lambda \vec{a}$ *represents the unit vector in the direction of* $\vec{a}$. We write it as

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

Note For any scalar k, $k\vec{0} = \vec{0}$.

10.5.1 *Components of a vector*

Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the x-axis, y-axis and z-axis, respectively. Then, clearly

$$|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1 \text{ and } |\overrightarrow{OC}| = 1$$

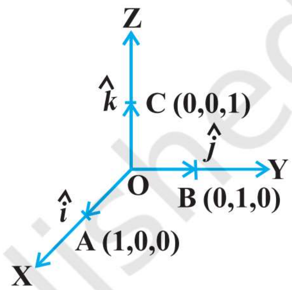


Fig 10.13

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called *unit vectors along the axes* OX, OY and OZ, respectively, and denoted by $\hat{i}, \hat{j}$ and $\hat{k}$, respectively (Fig 10.13).

Now, consider the position vector $\overrightarrow{OP}$ of a point P(x, y, z) as in Fig 10.14. Let P_1 be the foot of the perpendicular from P on the plane XOY.

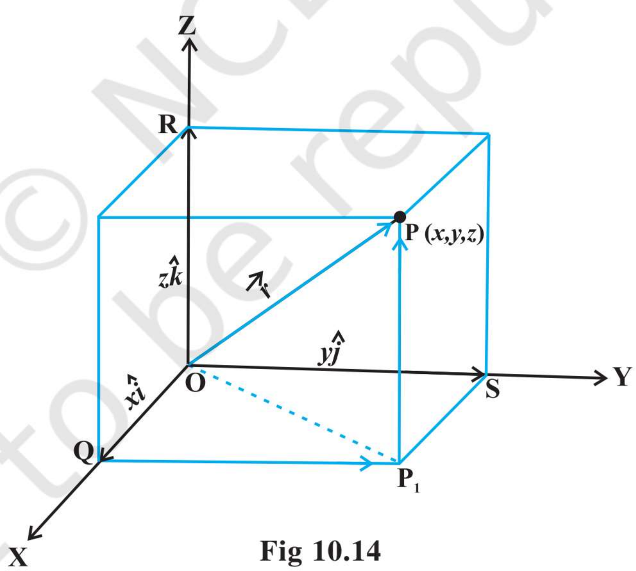


Fig 10.14

We, thus, see that P_1P is parallel to z-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the x, y and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$

and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its ***component form.*** Here, x, y and z are called as the ***scalar components*** of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the ***vector components*** of $\vec{r}$ along the respective axes. Sometimes x, y and z are also termed as ***rectangular components***.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle OQP_1 (Fig 10.14)

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2},$$

and in the right angle triangle OP_1P, we have

$$\overrightarrow{OP} = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vector $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_1 = b_1,\ a_2 = b_2 \quad \text{and} \quad a_3 = b_3$$

(iv) the multiplication of vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$