How can we say that a given point lies on the given line? Its answer may be that for a given line we should have a definite condition on the points lying on the line. Suppose P $(x, y)$ is an arbitrary point in the XY-plane and L is the given line. For the equation of L, we wish to construct a *statement* or *condition* for the point P that is true, when P is on L, otherwise false. Of course the statement is merely an algebraic equation involving the variables $x$ and $y$. Now, we will discuss the equation of a line under different conditions.

**10.3.1** ***Horizontal and vertical lines*** If a horizontal line L is at a distance $a$ from the $x$-axis then ordinate of every point lying on the line is either $a$ or $-a$ [Fig 10.11 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the $y$-axis. Similarly, the equation of a vertical line at a distance $b$ from the $y$-axis is either $x = b$ or $x = -b$ [Fig 10.11(b)].

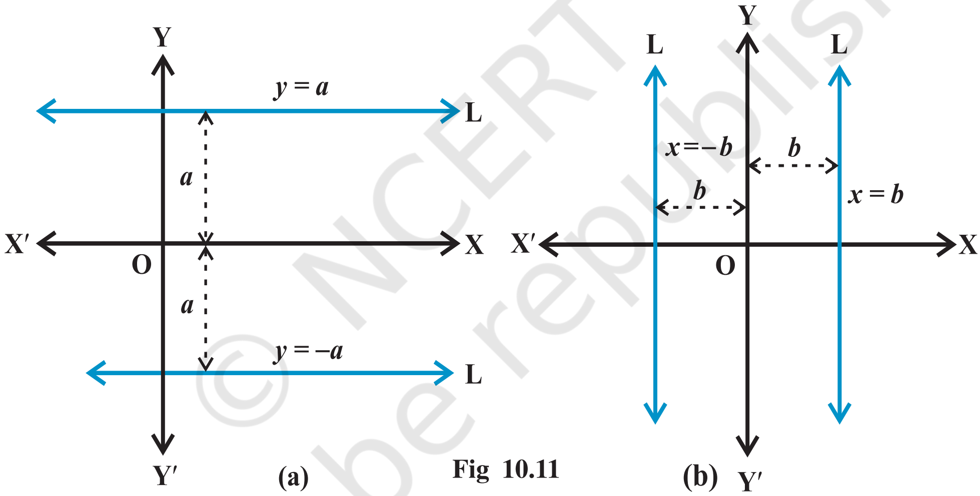

Fig 10.11

**Example 6** Find the equations of the lines parallel to axes and passing through $(-2, 3)$.

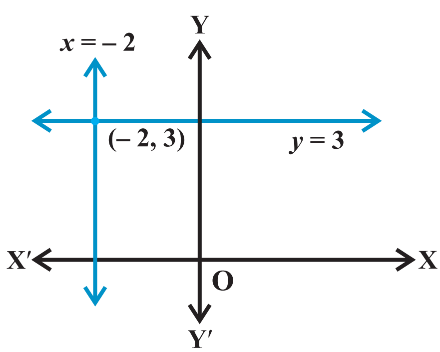

Fig 10.12

**Solution** Position of the lines is shown in the Fig 10.12. The $y$-coordinate of every point on the line parallel to $x$-axis is 3, therefore, equation of the line parallel to $x$-axis and passing through $(-2, 3)$ is $y = 3$. Similarly, equation of the line parallel to $y$-axis and passing through $(-2, 3)$ is $x = -2$.

**10.3.2 *Point-slope form*** Suppose that $P_0(x_0, y_0)$ is a fixed point on a non-vertical line L, whose slope is $m$. Let P $(x, y)$ be an arbitrary point on L (Fig 10.13).

Then, by the definition, the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \quad \text{...(1)}$$

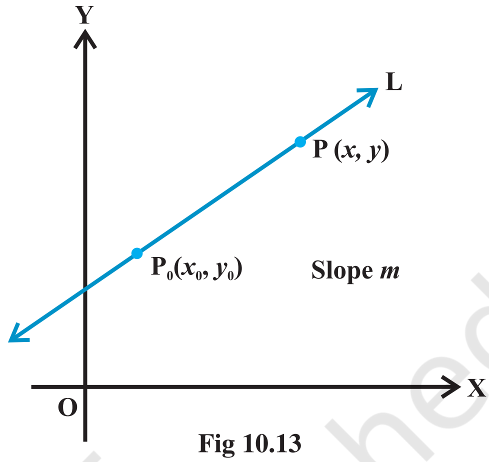

Fig 10.13

Since the point $P_0(x_0, y_0)$ along with all points $(x, y)$ on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

Thus, the point $(x, y)$ lies on the line with slope $m$ through the fixed point $(x_0, y_0)$, if and only if, its coordinates satisfy the equation

$$y - y_0 = m(x - x_0)$$

**Example 7** Find the equation of the line through (– 2, 3) with slope – 4.

**Solution** Here $m = -4$ and given point $(x_0, y_0)$ is (– 2, 3).

By slope-intercept form formula (1) above, equation of the given line is

$y - 3 = -4(x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

**10.3.3 *Two-point form*** Let the line L passes through two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let P $(x, y)$ be a general point on L (Fig 10.14).

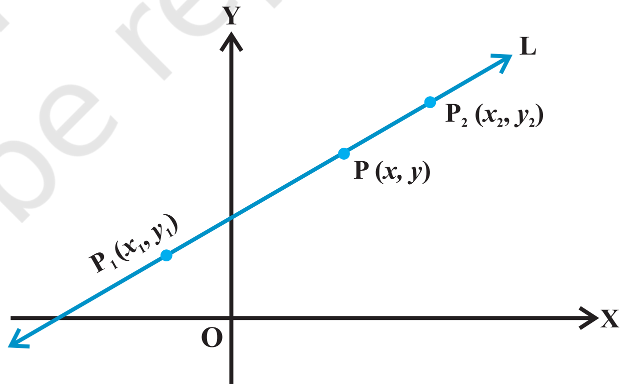

Fig 10.14

The three points $P_1$, $P_2$ and P are collinear, therefore, we have

slope of $P_1P$ = slope of $P_1P_2$

$$\text{i.e.,} \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{or} \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Thus, equation of the line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{... (2)}$$

**Example 8** Write the equation of the line through the points (1, –1) and (3, 5).

**Solution** Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5 - (-1)}{3 - 1}(x - 1)$$

or $\qquad -3x + y + 4 = 0$, which is the required equation.

**10.3.4 *Slope-intercept form*** Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines.

**Case I** Suppose a line L with slope $m$ cuts the $y$-axis at a distance $c$ from the origin (Fig10.15). The distance $c$ is called the *y-intercept* of the line L. Obviously, coordinates of the point where the line meet the $y$-axis are $(0, c)$. Thus, L has slope $m$ and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

$$y - c = m(x - 0) \quad \text{or} \quad y = mx + c$$

Thus, the point $(x, y)$ on the line with slope $m$ and $y$-intercept $c$ lies on the line if and only if

$$y = mx + c \qquad \text{...(3)}$$

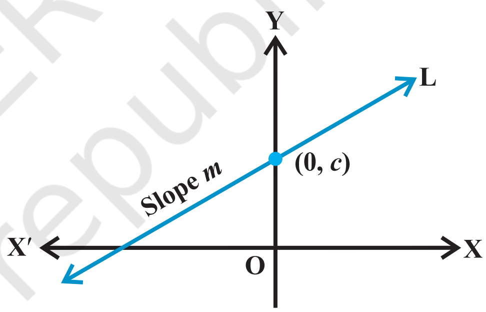

**Fig 10.15**

Note that the value of $c$ will be positive or negative according as the intercept is made on the positive or negative side of the $y$-axis, respectively.

**Case II** Suppose line L with slope $m$ makes $x$-intercept $d$. Then equation of L is

$$y = m(x - d) \qquad \text{... (4)}$$

Students may derive this equation themselves by the same method as in Case I.

**Example 9** Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where $\theta$ is the inclination of the line and (i) $y$-intercept is $-\frac{3}{2}$ (ii) $x$-intercept is 4.

**Solution** (i) Here, slope of the line is $m = \tan\theta = \frac{1}{2}$ and $y$ - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan\theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x-4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

**10.3.5 *Intercept - form*** Suppose a line L makes $x$-intercept $a$ and $y$-intercept $b$ on the axes. Obviously L meets $x$-axis at the point $(a, 0)$ and $y$-axis at the point $(0, b)$ (Fig .10.16). By two-point form of the equation of the line, we have

$$y - 0 = \frac{b-0}{0-a}(x-a) \quad \text{or} \quad ay = -bx + ab,$$

i.e., $$\frac{x}{a} + \frac{y}{b} = 1.$$

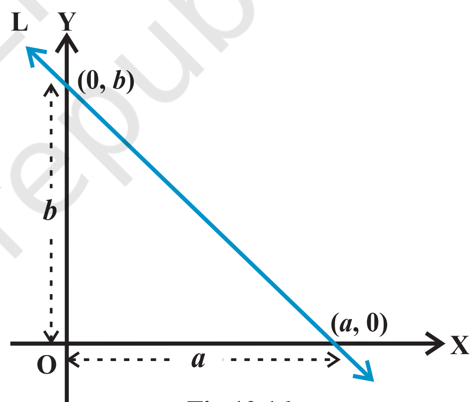


**Fig 10.16**

Thus, equation of the line making intercepts $a$ and $b$ on $x$-and $y$-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (5)}$$

**Example 10** Find the equation of the line, which makes intercepts –3 and 2 on the $x$- and $y$-axes respectively.

**Solution** Here $a = -3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3} + \frac{y}{2} = 1 \quad \text{or} \quad 2x - 3y + 6 = 0.$$

**10.3.6 *Normal form*** Suppose a non-vertical line is known to us with following data:

(i) Length of the perpendicular (normal) from origin to the line.

(ii) Angle which normal makes with the positive direction of $x$-axis.

Let L be the line, whose perpendicular distance from origin O be OA = $p$ and the angle between the positive $x$-axis and OA be $\angle XOA = \omega$. The possible positions of line L in the Cartesian plane are shown in the Fig 10.17. Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the $x$-axis in each case.

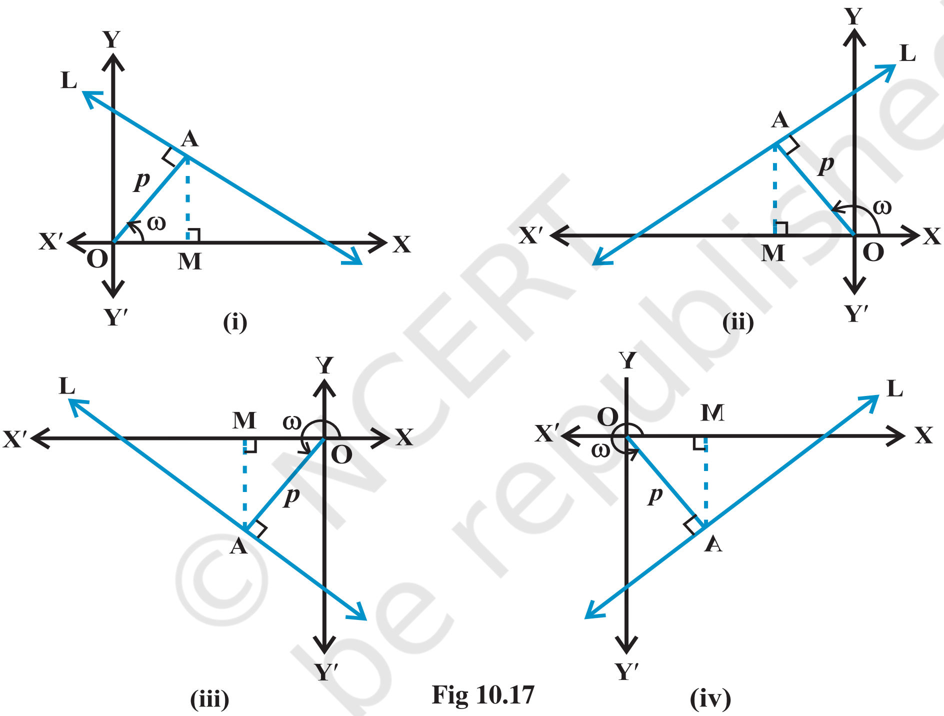


Fig 10.17



In each case, we have OM = $p \cos \omega$ and MA = $p \sin \omega$, so that the coordinates of the point A are ($p \cos \omega$, $p \sin \omega$).

Further, line L is perpendicular to OA. Therefore

$$\text{The slope of the line L} = -\frac{1}{\text{slope of OA}} = -\frac{1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Thus, the line L has slope $-\dfrac{\cos\omega}{\sin \omega}$ and point A $(p\cos\omega, p\sin \omega)$ on it. Therefore, by point-slope form, the equation of the line L is

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \quad \text{or} \quad x\cos\omega + y\sin\omega = p(\sin^2\omega + \cos^2\omega)$$

or $$x \cos \omega + y \sin \omega = p.$$

Hence, the equation of the line having normal distance $p$ from the origin and angle $\omega$ which the normal makes with the positive direction of $x$-axis is given by

$$x \cos \omega + y \sin \omega = p \qquad \text{... (6)}$$

**Example 11** Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of $x$-axis is 15°.

**Solution** Here, we are given $p = 4$ and $\omega = 15^0$ (Fig10.18).

Now $$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

and $$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{(Why?)}$$

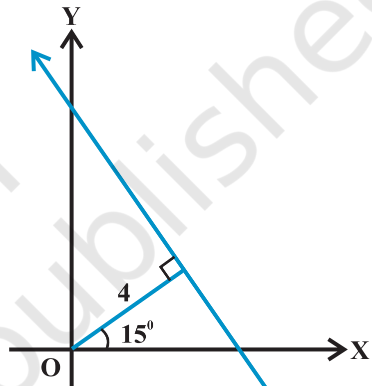

**Fig 10.18**

By the normal form (6) above, the equation of the line is

$$x\cos 15^0 + y\sin 15^0 = 4 \text{ or } \frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}-1}{2\sqrt{2}}y = 4 \quad \text{or} \quad \left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y = 8\sqrt{2}.$$

This is the required equation.

**Example 12** The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.

**Solution** Assuming F along $x$-axis and K along $y$-axis, we have two points (32, 273) and (212, 373) in XY-plane. By two-point form, the point (F, K) satisfies the equation

$$\text{K} - 273 = \frac{373-273}{212-32}(\text{F}-32) \quad \text{or} \quad \text{K} - 273 = \frac{100}{180}(\text{F}-32)$$

or $$\text{K} = \frac{5}{9}(\text{F}-32) + 273 \qquad \text{... (1)}$$

which is the required relation.

When K = 0, Equation (1) gives

$$0 = \frac{5}{9}(F - 32) + 273 \quad \text{or} \quad F - 32 = -\frac{273 \times 9}{5} = -\ 491.4 \quad \text{or} \quad F = -459.4 .$$

***Alternate method*** We know that simplest form of the equation of a line is $y = mx + c$. Again assuming F along *x-axis* and K along *y-axis*, we can take equation in the form

$$K = mF + c \qquad \text{... (1)}$$

Equation (1) is satisfied by (32, 273) and (212, 373). Therefore

$$273 = 32m + c \qquad \text{... (2)}$$

and

$$373 = 212m + c \qquad \text{... (3)}$$

Solving (2) and (3), we get

$$m = \frac{5}{9} \text{ and } c = \frac{2297}{9}.$$

Putting the values of $m$ and $c$ in (1), we get

$$K = \frac{5}{9}F + \frac{2297}{9} \qquad \text{... (4)}$$

which is the required relation. When K = 0, (4) gives F = – 459.4.

**Note** We know, that the equation $y = mx + c$, contains two constants, namely, $m$ and $c$. For finding these two constants, we need two conditions satisfied by the equation of line. In all the examples above, we are given two conditions to determine the equation of the line.

## EXERCISE 10.2

In Exercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for the $x$-and $y$-axes.
2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.
3. Passing through (0, 0) with slope $m$.
4. Passing through $(2, 2\sqrt{3})$ and inclined with the $x$-axis at an angle of 75°.
5. Intersecting the $x$-axis at a distance of 3 units to the left of origin with slope –2.
6. Intersecting the $y$-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the $x$-axis.
7. Passing through the points (–1, 1) and (2, – 4).

8. Perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive $x$-axis is $30^0$.
9. The vertices of $\Delta$ PQR are P (2, 1), Q (–2, 3) and R (4, 5). Find equation of the median through the vertex R.
10. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).
11. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: $n$. Find the equation of the line.
12. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).
13. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.
14. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive $x$-axis. Also, find the equation of line parallel to it and crossing the $y$-axis at a distance of 2 units below the origin.
15. The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line.
16. The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C.
17. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?
18. P ($a$, $b$) is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a}+\frac{y}{b}=2$.
19. Point R ($h$, $k$) divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
20. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

## 10.4 General Equation of a Line

In earlier classes, we have studied general equation of first degree in two variables, A$x$ + B$y$ + C = 0, where A, B and C are real constants such that A and B are not zero simultaneously. Graph of the equation A$x$ + B$y$ + C = 0 is always a straight line.

Therefore, any equation of the form A$x$ + B$y$ + C = 0, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

**10.4.1 *Different forms of* Ax + By + *C* = 0** The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

**(a) *Slope-intercept form*** If B ≠ 0, then A$x$ + B$y$ + C = 0 can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{or} \quad y = mx + c \qquad \text{... (1)}$$

where $\quad m = -\frac{A}{B}$ and $c = -\frac{C}{B}$ .

We know that Equation (1) is the slope-intercept form of the equation of a line whose slope is $-\frac{A}{B}$, and $y$-intercept is $-\frac{C}{B}$.

If B = 0, then $x = -\frac{C}{A}$ ,which is a vertical line whose slope is undefined and $x$-intercept is $-\frac{C}{A}$.

**(b) Intercept form** If C ≠ 0, then A$x$ + B$y$ + C = 0 can be written as

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (2)}$$

where $\quad a = -\frac{C}{A}$ and $b = -\frac{C}{B}$ .

We know that equation (2) is intercept form of the equation of a line whose $x$-intercept is $-\frac{C}{A}$ and $y$-intercept is $-\frac{C}{B}$.

If C = 0, then A$x$ + B$y$ + C = 0 can be written as A$x$ + B$y$ = 0, which is a line passing through the origin and, therefore, has zero intercepts on the axes.

**(c) Normal form** Let $x \cos \omega + y \sin \omega = p$ be the normal form of the line represented by the equation A$x$ + B$y$ + C = 0 or A$x$ + B$y$ = – C. Thus, both the equations are same and therefore, $\quad \frac{A}{\cos \omega} = \frac{B}{\sin \omega} = -\frac{C}{p}$

which gives $$\cos\omega = -\frac{Ap}{C} \text{ and } \sin\omega = -\frac{Bp}{C}.$$

Now $$\sin^2\omega + \cos^2\omega = \left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or $$p^2 = \frac{C^2}{A^2+B^2} \quad \text{or} \quad p = \pm\frac{C}{\sqrt{A^2+B^2}}$$

Therefore $$\cos\omega = \pm\frac{A}{\sqrt{A^2+B^2}} \text{ and } \sin\omega = \pm\frac{B}{\sqrt{A^2+B^2}}.$$

Thus, the normal form of the equation A$x$ + B$y$ + C = 0 is

$$x\cos\omega + y\sin\omega = p,$$

where $\cos\omega = \pm\dfrac{A}{\sqrt{A^2+B^2}}$, $\sin\omega = \pm\dfrac{B}{\sqrt{A^2+B^2}}$ and $p = \pm\dfrac{C}{\sqrt{A^2+B^2}}$.

Proper choice of signs is made so that $p$ should be positive.

**Example 13** Equation of a line is $3x - 4y + 10 = 0$. Find its (i) slope, (ii) $x$ - and $y$-intercepts.

**Solution** (i) Given equation $3x - 4y + 10 = 0$ can be written as

$$y = \frac{3}{4}x + \frac{5}{2} \qquad \text{... (1)}$$

Comparing (1) with $y = mx + c$, we have slope of the given line as $m = \dfrac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as

$$3x - 4y = -10 \quad \text{or} \quad \frac{x}{-\frac{10}{3}} + \frac{y}{\frac{5}{2}} = 1 \qquad \text{... (2)}$$

Comparing (2) with $\dfrac{x}{a} + \dfrac{y}{b} = 1$, we have $x$-intercept as $a = -\dfrac{10}{3}$ and $y$-intercept as $b = \dfrac{5}{2}$.

**Example 14** Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of $p$ and ω.

**Solution** Given equation is

$$\sqrt{3}x + y - 8 = 0 \qquad \text{... (1)}$$

Dividing (1) by $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$, we get

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \text{ or } \cos 30°\, x + \sin 30°\, y = 4 \qquad \text{... (2)}$$

Comparing (2) with $x \cos \omega + y \sin \omega = p$, we get $p = 4$ and $\omega = 30°$.

**Example15** Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

**Solution** Given lines are

$$y - \sqrt{3}x - 5 = 0 \text{ or } y = \sqrt{3}x + 5 \qquad \text{... (1)}$$

and

$$\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3} \qquad \text{... (2)}$$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{...}$$

(3)

Putting the values of $m_1$ and $m_2$ in (3), we get

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}}\right| = \left|\frac{1-3}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

which gives $\theta = 30°$. Hence, angle between two lines is either 30° or 180° – 30° = 150°.

**Example 16** Show that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $b_1, b_2 \neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and (ii) Perpendicular if $a_1 a_2 + b_1 b_2 = 0$.

**Solution** Given lines can be written as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \qquad \ldots (1)$$

and

$$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \qquad \ldots (2)$$

Slopes of the lines (1) and (2) are $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$, respectively. Now

(i) Lines are parallel, if $m_1 = m_2$, which gives

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

(ii) Lines are perpendicular, if $m_1.m_2 = -1$, which gives

$$\frac{a_1}{b_1}.\frac{a_2}{b_2} = -1 \text{ or } a_1a_2 + b_1b_2 = 0$$

**Example 17** Find the equation of a line perpendicular to the line $x - 2y + 3 = 0$ and passing through the point (1, – 2).

**Solution** Given line $x - 2y + 3 = 0$ can be written as

$$y = \frac{1}{2}x + \frac{3}{2} \qquad \ldots(1)$$

Slope of the line (1) is $m_1 = \frac{1}{2}$. Therefore, slope of the line perpendicular to line (1) is

$$m_2 = -\frac{1}{m_1} = -2$$

Equation of the line with slope – 2 and passing through the point (1, – 2) is

$$y - (-2) = -2(x - 1) \quad \text{or} \quad y = -2x,$$

which is the required equation.