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Chapter 10

STRAIGHT LINES

❖ *Geometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit. – H. FREUDENTHAL* ❖

10.1 Introduction

René Descartes
(1596 -1650)

We are familiar with two-dimensional *coordinate geometry* from earlier classes. Mainly, it is a combination of *algebra* and *geometry*. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book 'La Géométry, published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as *analytical geometry*. In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formulae, etc. All these concepts are the basics of coordinate geometry.

Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 10.1.

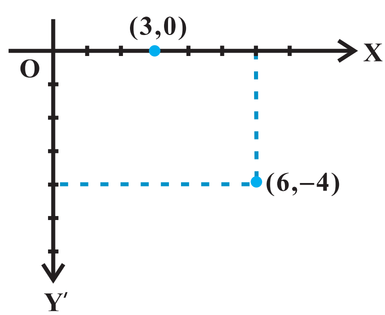

Fig 10.1

We may note that the point (6, – 4) is at 6 units distance from the *y*-axis measured along the positive *x*-axis and at 4 units distance from the *x*-axis measured along the negative *y*-axis. Similarly, the point (3, 0) is at 3 units distance from the *y*-axis measured along the positive *x*-axis and has zero distance from the *x*-axis.

We also studied there following important formulae:

I. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, distance between the points (6, – 4) and (3, 0) is

$$\sqrt{(3-6)^2 + (0+4)^2} = \sqrt{9+16} = 5 \text{ units.}$$

II. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m: n$ are $\left(\dfrac{m\,x_2 + n\,x_1}{m+n}, \dfrac{m\,y_2 + n\,y_1}{m+n}\right)$.

For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by $x = \dfrac{1.(-3)+3.1}{1+3} = 0$

and $y = \dfrac{1.9 + 3.(-3)}{1+3} = 0.$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

IV. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right| .$$

For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is

$$\frac{1}{2}\left|4(-2-16)+3(16-4)+(-3)(4+2)\right| = \frac{|-54|}{2} = 27.$$

Remark If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

In the this Chapter, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – *straight line*. Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which *slope* is most essential.

10.2 Slope of a Line

A line in a coordinate plane forms two angles with the x-axis, which are supplementary.

The angle (say) θ made by the line l with positive direction of x-axis and measured anti clockwise is called the *inclination of the line*. Obviously $0° \le \theta \le 180°$ (Fig 10.2).

We observe that lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°.

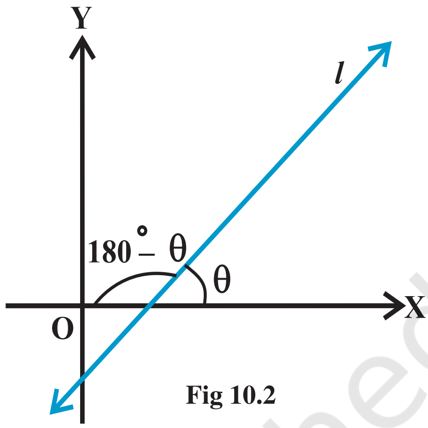

Fig 10.2

Definition 1 If θ is the inclination of a line l, then tan θ is called the *slope* or *gradient* of the line l.

The slope of a line whose inclination is 90° is not defined.

The slope of a line is denoted by m.

Thus, $m = \tan \theta,\ \theta \ne 90°$

It may be observed that the slope of x-axis is zero and slope of y-axis is not defined.

10.2.1 *Slope of a line when coordinates of any two points on the line are given*

We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non-vertical line l whose inclination is θ. Obviously, $x_1 \ne x_2$, otherwise the line will become perpendicular to x-axis and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take these two cases.

Draw perpendicular QR to x-axis and PM perpendicular to RQ as shown in Figs. 10.3 (i) and (ii).

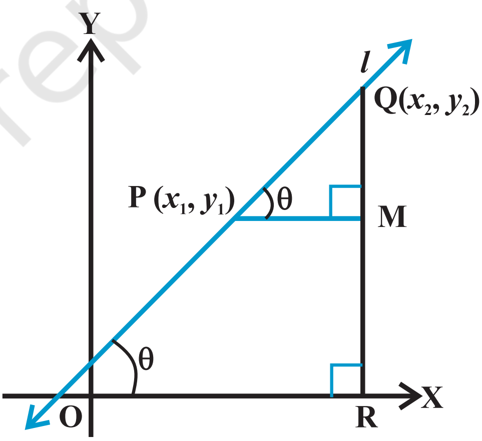

Fig 10. 3 (i)

Case 1 When angle θ is acute:

In Fig 10.3 (i), $\angle MPQ = \theta$. ... (1)

Therefore, slope of line $l = m = \tan \theta$.

But in ΔMPQ, we have $\tan\theta = \dfrac{MQ}{MP} = \dfrac{y_2 - y_1}{x_2 - x_1}$. ... (2)

From equations (1) and (2), we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Case II When angle θ is obtuse:
In Fig 10.3 (ii), we have
$\angle MPQ = 180° - \theta$.

Therefore, $\theta = 180° - \angle MPQ$.

Now, slope of the line l

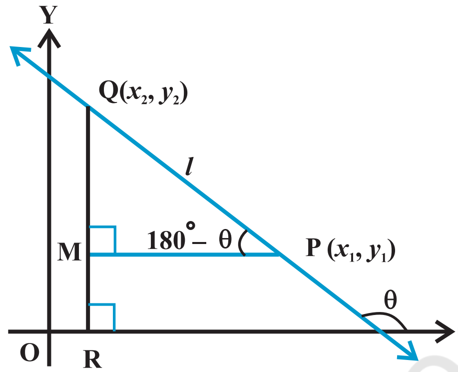

Fig 10. 3 (ii)

$$m = \tan\theta$$
$$= \tan(180° - \angle MPQ) = -\tan \angle MPQ$$
$$= -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consequently, we see that in both the cases the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

10.2.2 ***Conditions for parallelism and perpendicularity of lines in terms of their slopes*** In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

If the line l_1 is parallel to l_2 (Fig 10.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

$$m_1 = m_2.$$

Then $\tan\alpha = \tan\beta$.

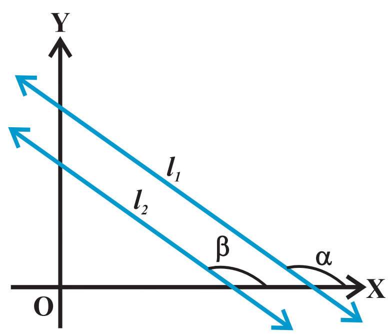

Fig 10. 4

By the property of tangent function (between 0° and 180°), α = β.

Therefore, the lines are parallel.

Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular (Fig 10.5), then $\beta = \alpha + 90°$.

Therefore, $\tan \beta = \tan (\alpha + 90°)$

$$= -\cot \alpha = -\frac{1}{\tan\alpha}$$

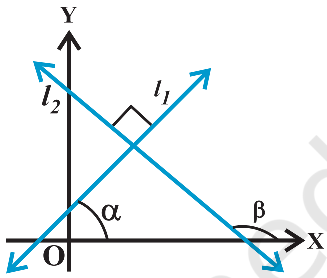

Fig 10. 5

i.e., $$m_2 = -\frac{1}{m_1} \quad \text{or} \quad m_1\, m_2 = -1$$

Conversely, if $m_1\, m_2 = -1$, i.e., $\tan \alpha \tan \beta = -1$.

Then $\tan \alpha = -\cot \beta = \tan (\beta + 90°)$ or $\tan (\beta - 90°)$

Therefore, α and β differ by 90°.

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $$m_2 = -\frac{1}{m_1} \text{ or, } m_1\, m_2 = -1.$$

Let us consider the following example.

Example 1 Find the slope of the lines:

(a) Passing through the points (3, – 2) and (–1, 4),

(b) Passing through the points (3, – 2) and (7, – 2),

(c) Passing through the points (3, – 2) and (3, 4),

(d) Making inclination of 60° with the positive direction of x-axis.

Solution (a) The slope of the line through (3, – 2) and (– 1, 4) is

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}.$$

(b) The slope of the line through the points (3, – 2) and (7, – 2) is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

(c) The slope of the line through the points (3, – 2) and (3, 4) is

$$m = \frac{4-(-2)}{3-3} = \frac{6}{0}, \text{ which is not defined.}$$

(d) Here inclination of the line $\alpha = 60°$. Therefore, slope of the line is $m = \tan 60° = \sqrt{3}$.

10.2.3 *Angle between two lines* When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2.$$

We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig10.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \dfrac{m_2-m_1}{1+m_1m_2}$ (as $1 + m_1m_2 \neq 0$)

and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2-m_1}{1+m_1m_2}, \text{ as } 1 + m_1m_2 \neq 0$$

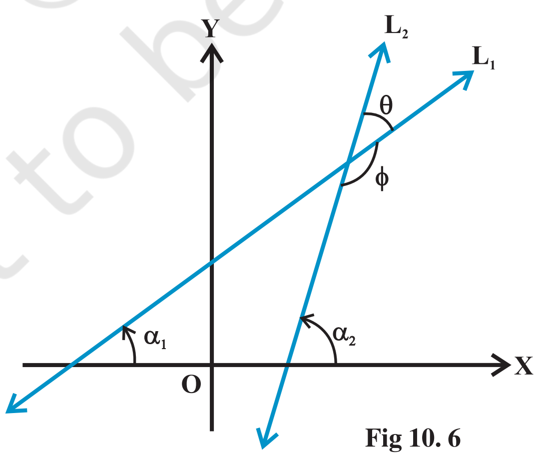

Fig 10. 6

Now, there arise two cases:

Case I If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then tan θ will be positive and tan φ will be negative, which means θ will be acute and φ will be obtuse.

Case II If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is negative, then tan θ will be negative and tan φ will be positive, which means that θ will be obtuse and φ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ as } 1 + m_1 m_2 \neq 0 \qquad \text{... (1)}$$

The obtuse angle (say φ) can be found by using $\phi = 180^0 - \theta$.

Example 2 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{... (1)}$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$.

Now, putting these values in (1), we get

$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \text{ or } 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|,$$

which gives

$$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = 1 \text{ or } \frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = -1.$$

Therefore $m = 3$ or $m = -\frac{1}{3}$.

Hence, slope of the other line is 3 or $-\frac{1}{3}$. Fig 10.7 explains the reason of two answers.

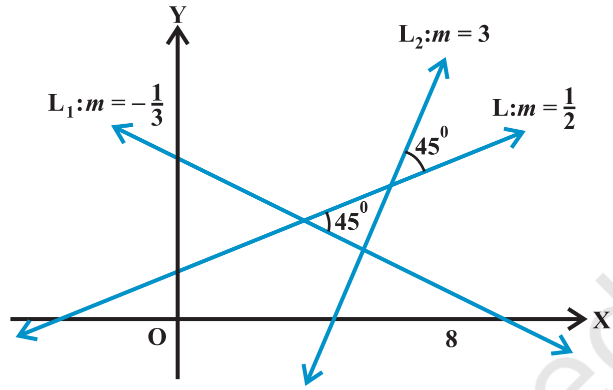

Fig 10. 7

Example 3 Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Solution Slope of the line through the points (– 2, 6) and (4, 8) is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points (8, 12) and (x, 24) is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,
$m_1\, m_2 = -1$, which gives

$$\frac{1}{3} \times \frac{12}{x-8} = -1 \text{ or } x = 4.$$

10.2.4 *Collinearity of three points*

We know that slopes of two parallel lines are equal. If two lines having the same slope pass through a common point, then two lines will coincide. Hence, if A, B and C are three points in the XY-plane, then they will lie on a line, i.e., three points are collinear (Fig 10.8) if and only if slope of AB = slope of BC.

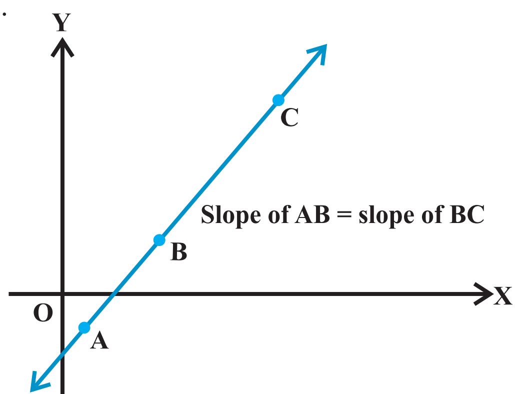

Fig 10. 8

Example 4 Three points P (h, k), Q (x_1, y_1) and R (x_2, y_2) lie on a line. Show that

$$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Solution Since points P, Q and R are collinear, we have

$$\text{Slope of PQ} = \text{Slope of QR, i.e., } \frac{y_1 - k}{x_1 - h} = \frac{y_2 - y_1}{x_2 - x_1}$$

or $$\frac{k - y_1}{h - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

or $$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Example 5 In Fig 10.9, time and distance graph of a linear motion is given. Two positions of time and distance are recorded as, when T = 0, D = 2 and when T = 3, D = 8. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

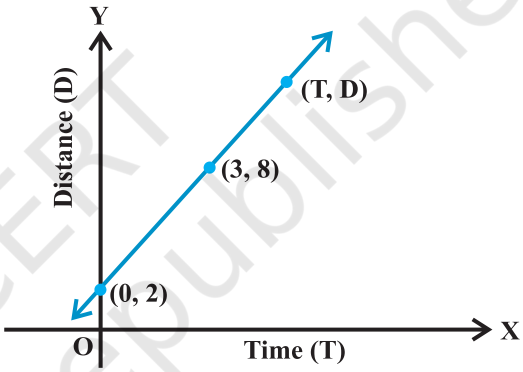

Fig 10.9

Solution Let (T, D) be any point on the line, where D denotes the distance at time T. Therefore, points (0, 2), (3, 8) and (T, D) are collinear so that

$$\frac{8-2}{3-0} = \frac{\text{D}-8}{\text{T}-3} \quad \text{or} \quad 6(\text{T}-3) = 3(\text{D}-8)$$

or $\text{D} = 2(\text{T} + 1)$,

which is the required relation.

EXERCISE 10.1

1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area.
2. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
3. Find the distance between P (x_1, y_1) and Q (x_2, y_2) when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.
4. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).
5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).

6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.
7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.
8. Find the value of x for which the points $(x, -1)$, (2,1) and (4, 5) are collinear.
9. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.
10. Find the angle between the x-axis and the line joining the points (3,–1) and (4,–2).
11. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
12. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that
$$k - y_1 = m\,(h - x_1).$$
13. If three points $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
14. Consider the following population and year graph (Fig 10.10), find the slope of the line AB and using it, find what will be the population in the year 2010?

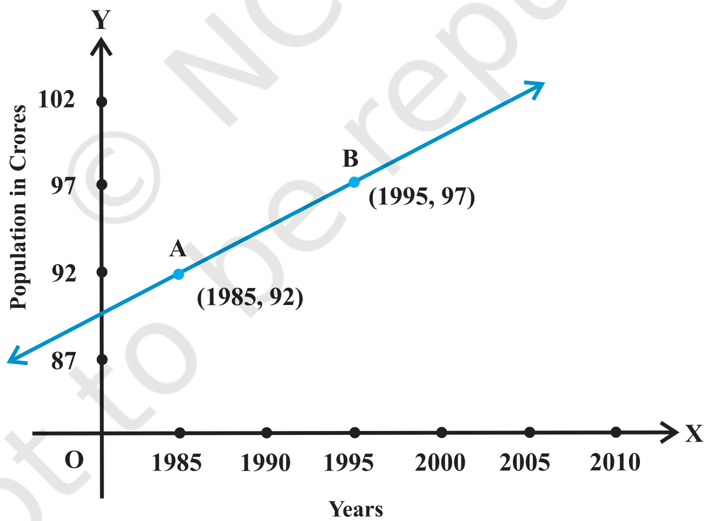

Fig 10.10

10.3 Various Forms of the Equation of a Line

We know that every line in a plane contains infinitely many points on it. This relationship between line and points leads us to find the solution of the following problem: