6.4 Graphical Solution of Linear Inequalities in Two Variables

In earlier section, we have seen that a graph of an inequality in one variable is a visual representation and is a convenient way to represent the solutions of the inequality. Now, we will discuss graph of a linear inequality in two variables.

We know that a line divides the Cartesian plane into two parts. Each part is known as a half plane. A vertical line will divide the plane in left and right half planes and a non-vertical line will divide the plane into lower and upper half planes (Figs. 6.3 and 6.4).

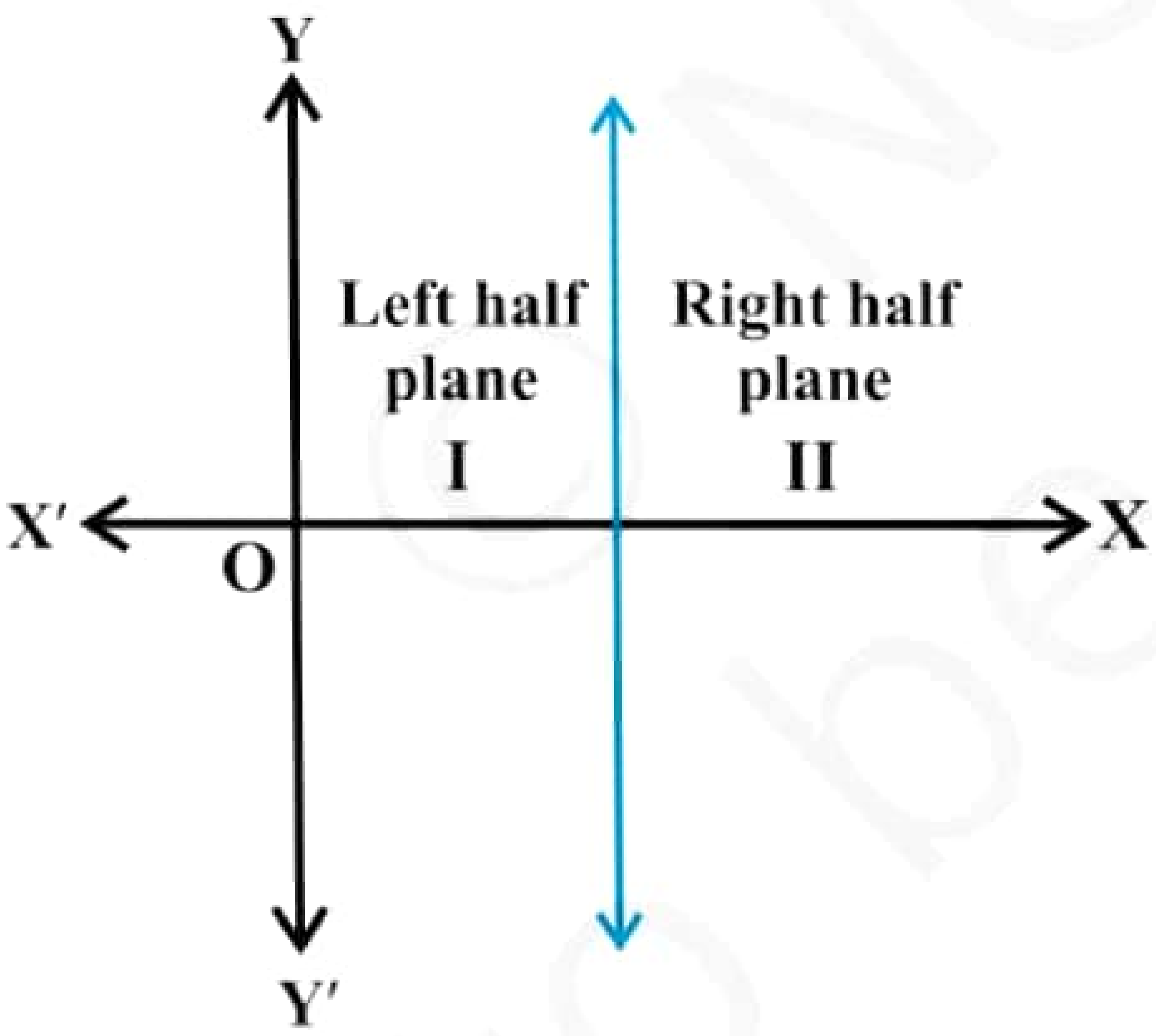

Fig 6.3

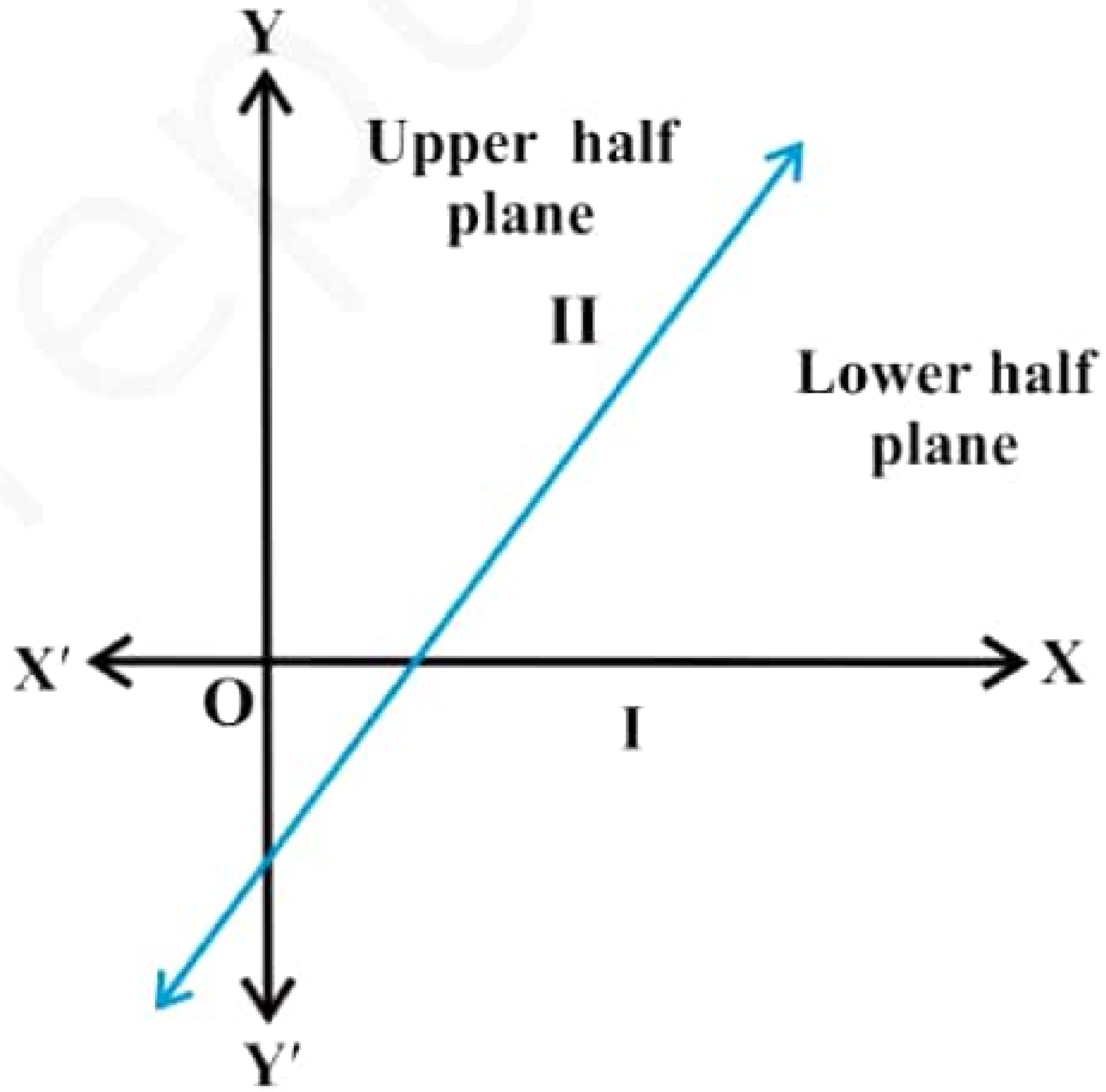

Fig 6.4

A point in the Cartesian plane will either lie on a line or will lie in either of the half planes I or II. We shall now examine the relationship, if any, of the points in the plane and the inequalities $ax + by < c$ or $ax + by > c$.

Let us consider the line

$$ax + by = c, \qquad a \neq 0, \quad b \neq 0 \qquad \dots (1)$$

There are three possibilities namely:

(i) $ax + by = c$ (ii) $ax + by > c$ (iii) $ax + by < c$.

In case (i), clearly, all points (x, y) satisfying (i) lie on the line it represents and conversely. Consider case (ii), let us first assume that $b > 0$. Consider a point P (α,β) on the line $ax + by = c$, $b > 0$, so that $a\alpha + b\beta = c$. Take an arbitrary point Q (α, γ) in the half plane II (Fig 6.5).

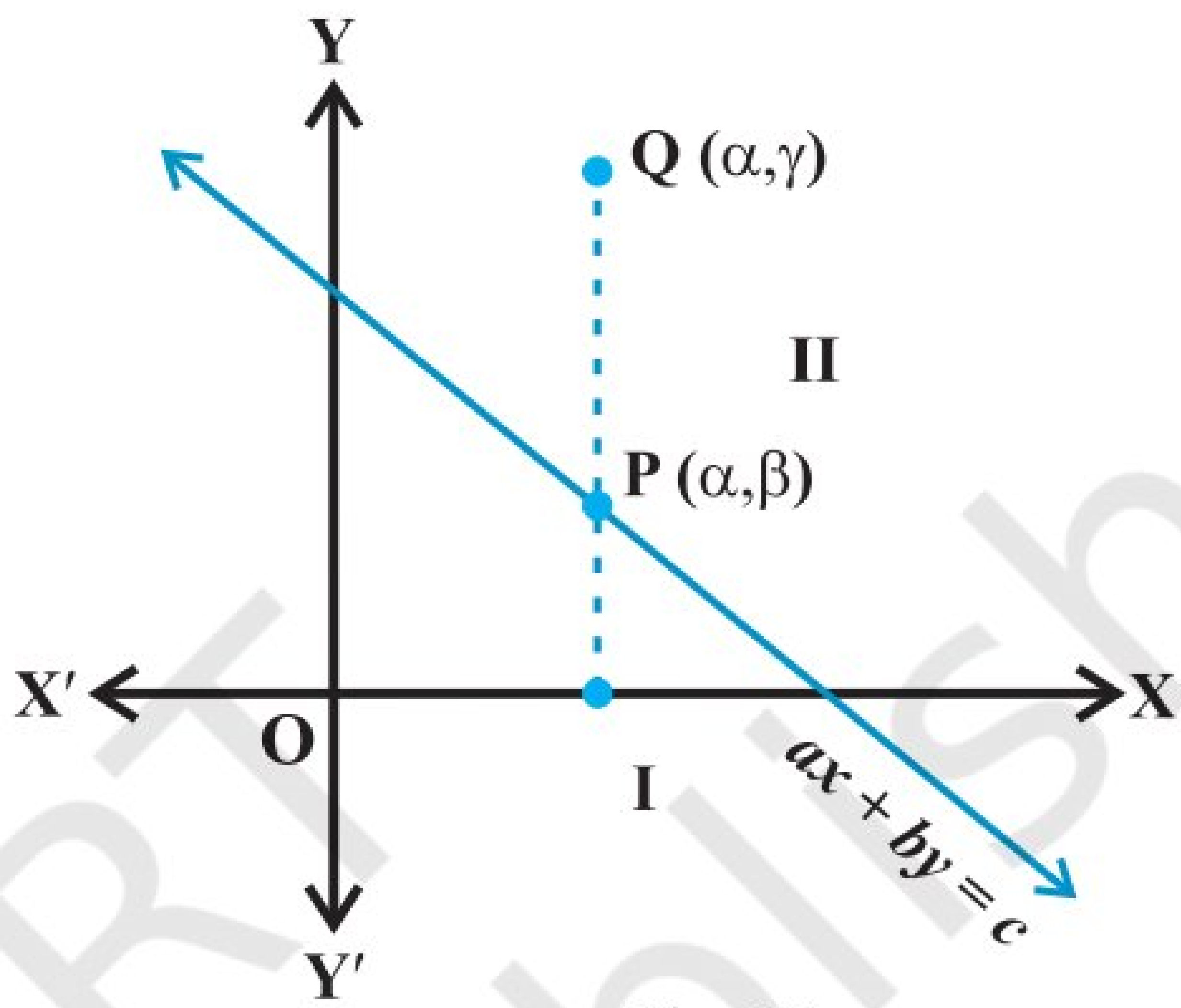

Fig 6.5

Now, from Fig 6.5, we interpret,

$\gamma > \beta$ (Why?)

or $b\gamma > b\beta$ or $a\alpha + b\gamma > a\alpha + b\beta$ (Why?)

or $a\alpha + b\gamma > c$

i.e., Q(α, γ) satisfies the inequality $ax + by > c$.

Thus, all the points lying in the half plane II above the line $ax + by = c$ satisfies the inequality $ax + by > c$. Conversely, let (α, β) be a point on line $ax + by = c$ and an arbitrary point Q(α, γ) satisfying

$$ax + by > c$$

so that $a\alpha + b\gamma > c$

$\Rightarrow$ $a\alpha + b\gamma > a\alpha + b\beta$ (Why?)

$\Rightarrow$ $\gamma > \beta$ (as $b > 0$)

This means that the point (α, γ) lies in the half plane II.

Thus, any point in the half plane II satisfies $ax + by > c$, and conversely any point satisfying the inequality $ax + by > c$ lies in half plane II.

In case $b < 0$, we can similarly prove that any point satisfying $ax + by > c$ lies in the half plane I, and conversely.

Hence, we deduce that all points satisfying $ax + by > c$ lies in one of the half planes II or I according as $b > 0$ or $b < 0$, and conversely.

Thus, graph of the inequality $ax + by > c$ will be one of the half plane (called *solution region*) and represented by shading in the corresponding half plane.

Note 1 The region containing all the solutions of an inequality is called the *solution region.*

2. In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) (not on line) and check whether it satisfies the inequality or not. If it satisfies, then the inequality represents the half plane and shade the region

which contains the point, otherwise, the inequality represents that half plane which does not contain the point within it. For convenience, the point (0, 0) is preferred.

3. If an inequality is of the type $ax + by \geq c$ or $ax + by \leq c$, then the points on the line $ax + by = c$ are also included in the solution region. So draw a dark line in the solution region.

4. If an inequality is of the form $ax + by > c$ or $ax + by < c$, then the points on the line $ax + by = c$ are not to be included in the solution region. So draw a broken or dotted line in the solution region.

In Section 6.2, we obtained the following linear inequalities in two variables *x and y:* $40x + 20y \leq 120$ ... (1)
while translating the word problem of purchasing of registers and pens by Reshma.

Let us now solve this inequality keeping in mind that x and y can be only whole numbers, since the number of articles cannot be a fraction or a negative number. In this case, we find the pairs of values of x and y, which make the statement (1) true. In fact, the set of such pairs will be the *solution set* of the inequality (1).

To start with, let $x = 0$. Then L.H.S. of (1) is

$$40x + 20y = 40\,(0) + 20y = 20y.$$

Thus, we have

$$20y \leq 120 \text{ or } y \leq 6 \qquad \text{... (2)}$$

For $x = 0$, the corresponding values of y can be 0, 1, 2, 3, 4, 5, 6 only. In this case, the solutions of (1) are (0, 0), (0, 1), (0,2), (0,3), (0,4), (0, 5) and (0, 6).

Similarly, other solutions of (1), when $x = 1$, 2 and 3 are: (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 0), (2, 1), (2, 2), (3, 0)
This is shown in Fig 6.6.

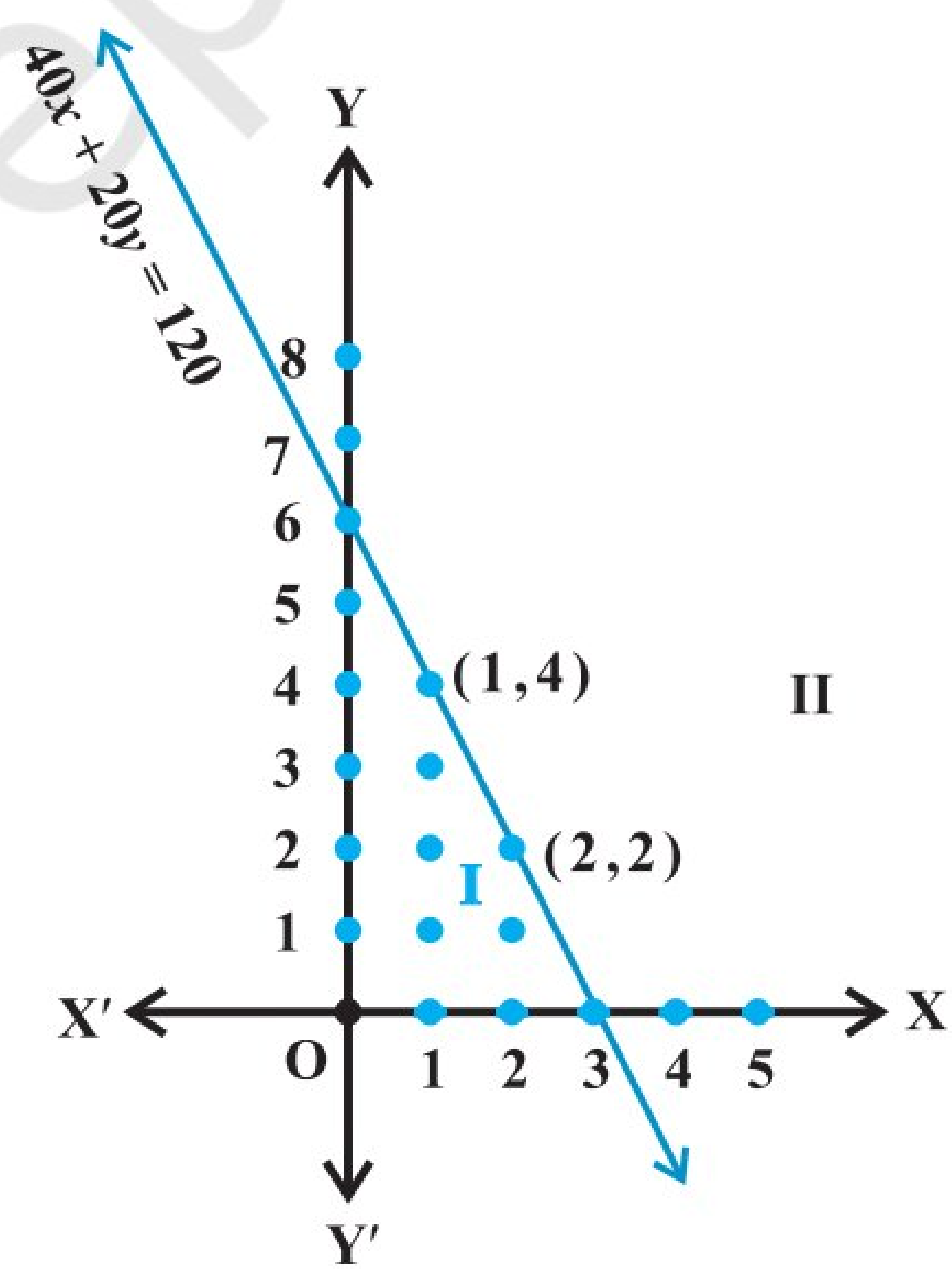

Fig 6.6

Let us now extend the domain of x and y from whole numbers to real numbers, and see what will be the solutions of (1) in this case. You will see that the graphical method of solution will be very convenient in this case. For this purpose, let us consider the (corresponding) equation and draw its graph.

$$40x + 20y = 120 \qquad \text{... (3)}$$

In order to draw the graph of the inequality (1), we take one point say (0, 0), in half plane I and check whether values of x and y satisfy the inequality or not.

We observe that $x = 0$, $y = 0$ satisfy the inequality. Thus, we say that the half plane I is the graph (Fig 6.7) of the inequality. Since the points on the line also satisfy the inequality (1) above, the line is also a part of the graph.

Thus, the graph of the given inequality is half plane I including the line itself. Clearly half plane II is not the part of the graph. Hence, *solutions* of inequality (1) will consist of all the points of its graph (half plane I including the line).

We shall now consider some examples to explain the above procedure for solving a linear inequality involving two variables.

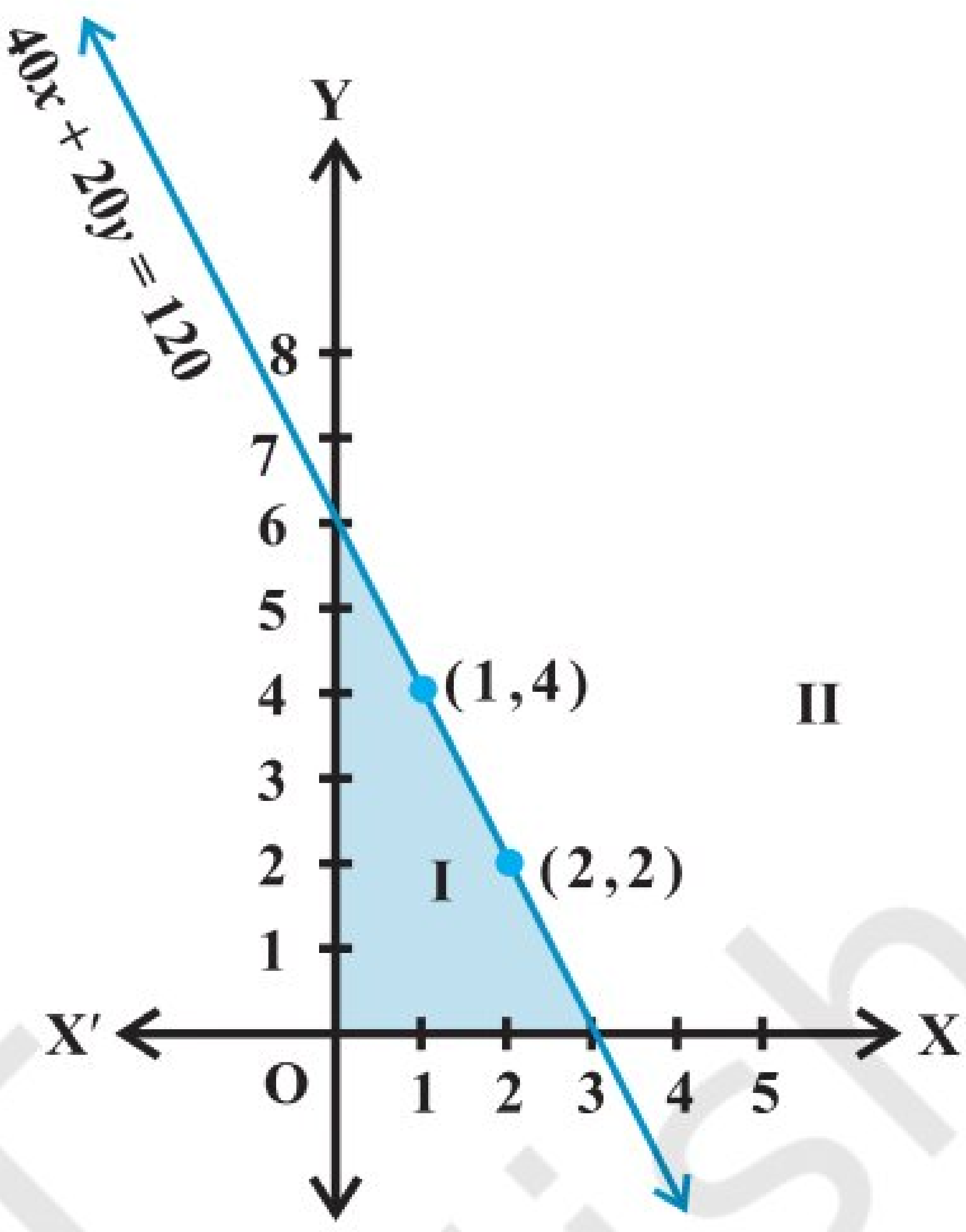

Fig 6.7

Example 9 Solve $3x + 2y > 6$ graphically.

Solution Graph of $3x + 2y = 6$ is given as dotted line in the Fig 6.8.

This line divides the *xy*-plane in two half planes I and II. We select a point (not on the line), say (0, 0), which lies in one of the half planes (Fig 6.8) and determine if this point satisfies the given inequality, we note that

$$3(0) + 2(0) > 6$$

or $\quad 0 > 6$, which is false.

Hence, half plane I is not the solution region of the given inequality. Clearly, any point on the line does not satisfy the given strict inequality. In other words, the shaded half plane II excluding the points on the line is the solution region of the inequality.

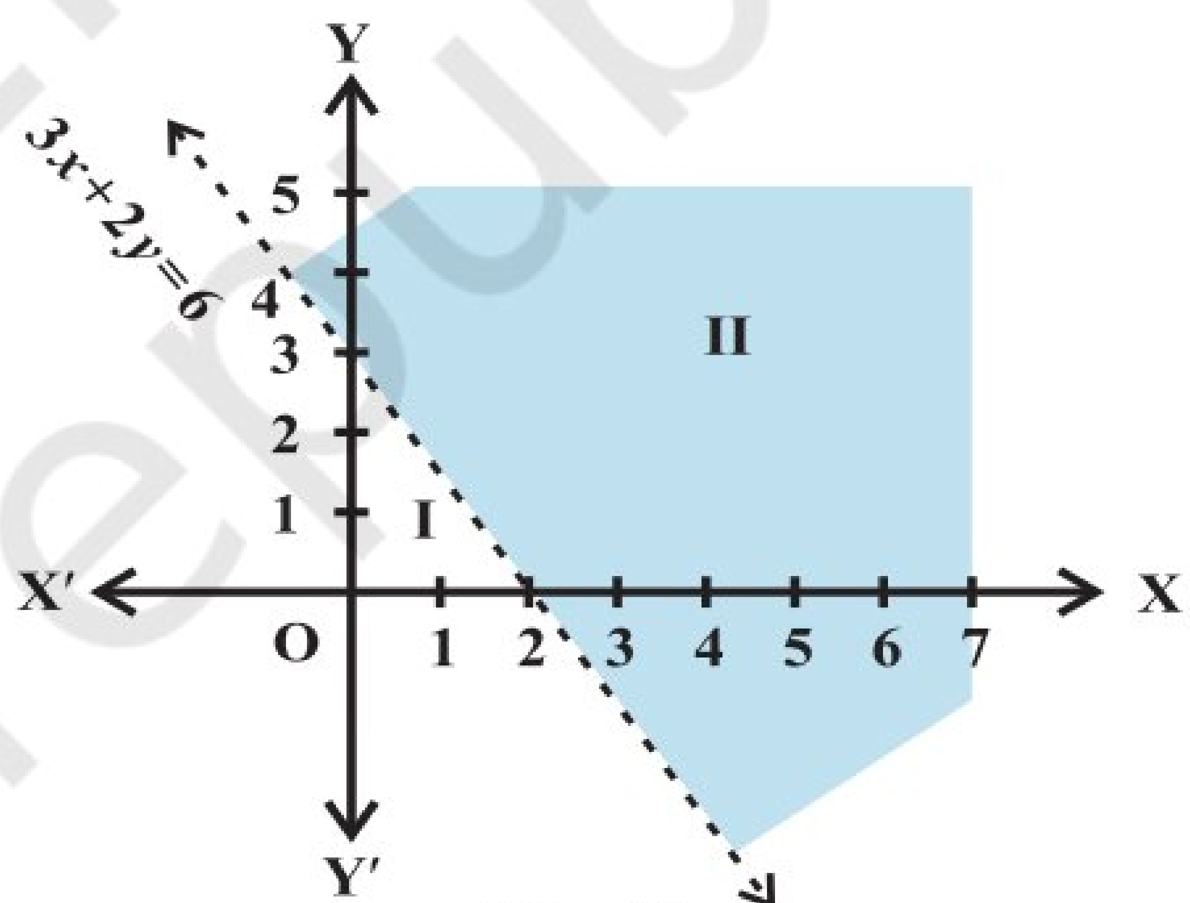

Fig 6.8

Example 10 Solve $3x - 6 \geq 0$ graphically in two dimensional plane.

Solution Graph of $3x - 6 = 0$ is given in the Fig 6.9.

We select a point, say (0, 0) and substituting it in given inequality, we see that:

$$3(0) - 6 \geq 0 \text{ or } -6 \geq 0 \text{ which is false.}$$

Thus, the solution region is the shaded region on the right hand side of the line $x = 2$.

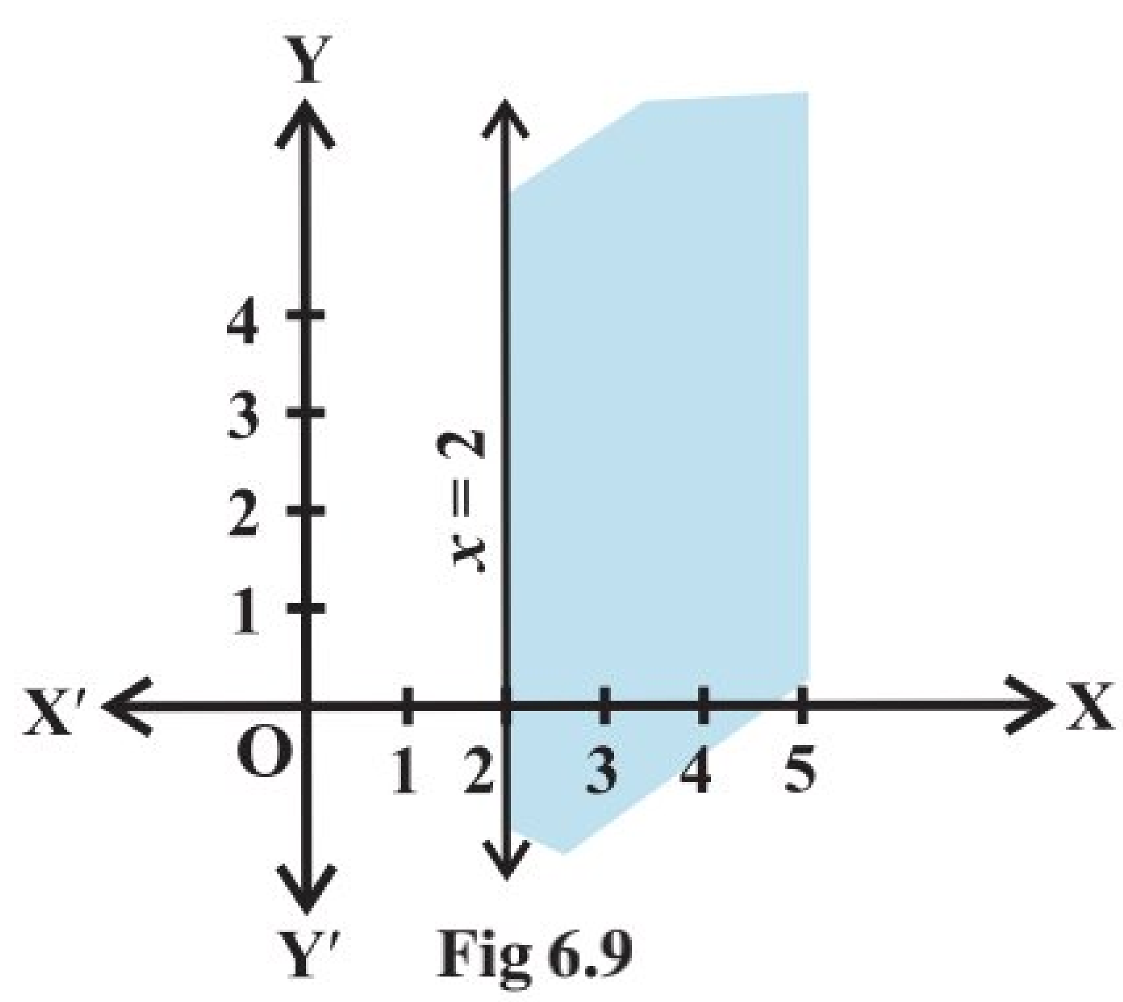

Fig 6.9

Example 11 Solve $y < 2$ graphically.

Solution Graph of $y = 2$ is given in the Fig 6.10.

Let us select a point, (0, 0) in lower half plane I and putting $y = 0$ in the given inequality, we see that

$$1 \times 0 < 2 \text{ or } 0 < 2 \text{ which is true.}$$

Thus, the solution region is the shaded region below the line $y = 2$. Hence, every point below the line (excluding all the points on the line) determines the solution of the given inequality.

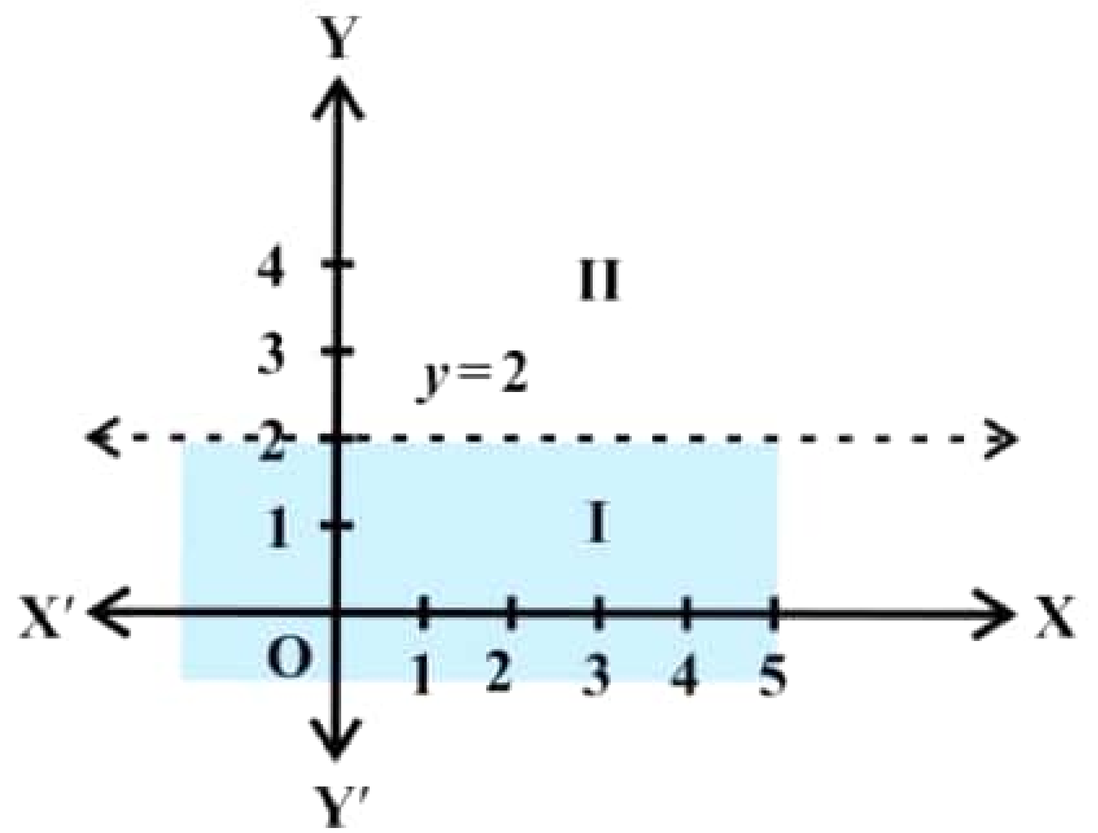

Fig 6.10

EXERCISE 6.2

Solve the following inequalities graphically in two-dimensional plane:

1. $x + y < 5$
2. $2x + y \geq 6$
3. $3x + 4y \leq 12$
4. $y + 8 \geq 2x$
5. $x - y \leq 2$
6. $2x - 3y > 6$
7. $-3x + 2y \geq -6$
8. $3y - 5x < 30$
9. $y < -2$
10. $x > -3$.