Example 26 Evaluate $\int_0^2 e^x\,dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x\,dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

Using the sum to n terms of a G.P., where $a = 1,\ r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x\,dx = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{2\,(e^2-1)}{\lim\limits_{n\to\infty}\left[\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right]\cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1\right]$$

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x\,dx$
2. $\int_0^5 (x+1)\,dx$
3. $\int_2^3 x^2\,dx$
4. $\int_1^4 (x^2 - x)\,dx$
5. $\int_{-1}^1 e^x\,dx$
6. $\int_0^4 (x + e^{2x})\,dx$

7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the area of the light shaded region

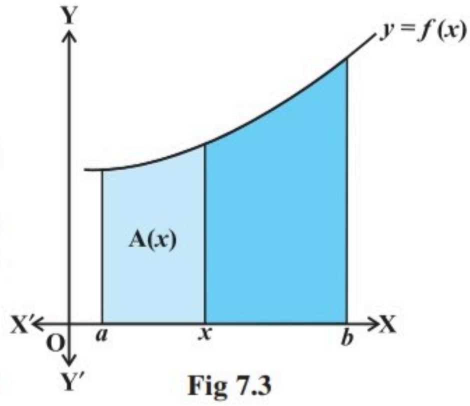

Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by A(x). We call the function A(x) as *Area function* and is given by

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \qquad \text{... (1)}$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function. Then $\mathbf{A}'(x) = f(x)$, for all $x \in [a, b]$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [\mathbf{F}(x)]_a^b = \mathbf{F}(b) - \mathbf{F}(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx$ = (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$.

For instance, the consideration of definite integral $\int_{-2}^{3} x(x^2 - 1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2 - 1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$\int_a^b f(x)\,dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$.

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate F(b) – F(a) = $[F(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\,dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx$

(iii) $\int_1^2 \frac{x\,dx}{(x+1)(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$

Solution

(i) Let $I = \int_2^3 x^2\,dx$. Since $\int x^2\,dx = \frac{x^3}{3} = F(x)$,

Therefore, by the second fundamental theorem, we get

$$I = F(3) - F(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $I = \int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

Thus, $\int \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30-x^{\frac{3}{2}})}\right] = F(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(9) - F(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $I = \int_1^2 \frac{x\,dx}{(x+1)(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $$\int \frac{x\,dx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = F(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(2) - F(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$. Consider $\int \sin^3 2t \cos 2t\,dt$

Put $\sin 2t = u$ so that $2\cos 2t\,dt = du$ or $\cos 2t\,dt = \frac{1}{2}\,du$

So $$\int \sin^3 2t \cos 2t\,dt = \frac{1}{2}\int u^3 du$$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = F(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$