Integration based on substitution methods:

Question 1:

Find the integration of

$$\int \frac{e^{tan^{-1}x}}{1+x^2}$$

Solution:

Given :

$$\int \frac{e^{tan^{-1}x}}{1+x^2}$$

Let t = $\tan^{-1}x$ ...... (1)

dt = (1/ 1+x^2) . dx

I = ∫ e^t . dt

= e^t + C ....(2)

Substituting the value of (1) in (2), we have I = $e^{\tan^{-1}x}$ + C. This is the required integration for the given function.

Question 2:

$$\int \frac{(x^2-1)}{x^3\sqrt{(2x^4-2x^2+1)}}dx \text{ is equal to}$$

(a) $\frac{\sqrt{(2x^4-2x^2+1)}}{x(x^2-1)}+c$

(b) $\frac{\sqrt{(2x^4-2x^2+1)}}{x^3}+c$

(c) $\frac{\sqrt{(2x^4-2x^2+1)}}{x^2}+c$

(d) $\frac{\sqrt{(2x^4-2x^2+1)}}{2x^2}+c$

Solution:

$$\text{Let } I=\int \frac{(x^2-1)dx}{x^3.x^2\sqrt{\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}$$

$$=\int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx}{\sqrt{\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}$$

$$\text{Substituting } 2-\frac{2}{x^2}+\frac{1}{x^4}=t$$

$$\therefore \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx=dt$$

$$\text{Then } I=\frac{1}{4}\int \frac{dt}{\sqrt{t}}=\frac{1}{4}.2\sqrt{t}+c$$

$$=\frac{\sqrt{(2x^4-2x^2+1)}}{2x^2}+c$$

Hence, the correct option is (d).

Question 3: (JEE Advanced 2011)

The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$ is

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$

(C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

Solution:

$$I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$$

Let $x^2 = t$

$$I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt$$

$$I = \frac{\ln 3 - \ln 2}{4} = \frac{1}{4}\ln\frac{3}{2}$$

Question 4: (MCQ)

$$\int \frac{\sqrt{4+x^2}}{x^6} dx = \frac{A(4+x^2)^{3/2}(Bx^2-6)}{x^5} + C,$$

then

(a) $A = \frac{1}{120}$ (b) $B = 1$

(c) $A = -\frac{1}{120}$ (d) $B = -1$

Solution:

Here, $I = \int \frac{\sqrt{4-x^2}}{x^6} dx = \int \frac{\sqrt{1+\frac{4}{x^2}}}{x^5} dx = \int \frac{\sqrt{1+\frac{4}{x^2}}}{x^2 \cdot x^3} dx$

Put $t = \sqrt{1+\frac{4}{x^2}} \Rightarrow t^2 = 1 + \frac{4}{x^2}$

$\therefore\ 2t\, dt = -\frac{8}{x^3} dx$

$$\Rightarrow \quad I = \frac{1}{16}\int (t^2 - t^4)\, dx = \frac{1}{16}\left\{\frac{t^3}{3} - \frac{t^5}{5}\right\} + C$$

$$= \frac{1}{120} \cdot \frac{(4+x^2)^{3/2}}{x^5}(x^2 - 6) + C$$

$$A = \frac{1}{120}, B = 1$$

Hence, (a) and (b) are the correct answers.

Question 5:

The value of the integral

$\int e^{\sin^2 x} (\cos x + \cos^3 x) \sin x \, dx$ *is*

(a) $\frac{1}{2} e^{\sin^2 x} (3 - \sin^2 x) + C$

(b) $e^{\sin^2 x} \left(1 + \frac{1}{2} \cos^2 x\right) + C$

(c) $e^{\sin^2 x} (3 \cos^2 x + 2\sin^2 x) + C$

(d) $e^{\sin^2 x} (2\cos^2 x + 3\sin^2 x) + C$

Solution:

Put $t = \sin^2 x$

The integral reduces to $I = \frac{1}{2} \int e^t (2 - t) \, dt = \frac{3}{2} e^t - \frac{te^t}{2} + C$

$= \frac{1}{2} e^{\sin^2 x}(3 - \sin^2 x) + C$ [option (a)]

$= e^{\sin^2 x} \left(1 + \frac{1}{2} \cos^2 x\right) + C$ [option (b)]

Hence, (a) and (b) are the correct answers.

Question 6:

The value of $\int \frac{dx}{\cos^6 x + \sin^6 x}$, *is equal to*

(a) $\tan^{-1}(2 \cot 2x) + C$ (b) $\tan^{-1}(\cot 2x) + C$

(c) $\tan^{-1}\left(\frac{1}{2} \cot 2x\right) + C$ (d) $\tan^{-1}(-2 \cot 2x) + C$

Solution:

Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x \, dx}{1 + \tan^6 x}$

$= \int \frac{(1 + \tan^2 x)^2 \cdot \sec^2 x}{1 + \tan^6 x} dx$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt = \int \frac{(1 + t^2)^2}{1 + t^6} dt$

$= \int \frac{(1 + t^2)^2}{(1 + t^2)(1 - t^2 + t^4)} dt$

$= \int \frac{1 + t^2}{1 - t^2 + t^4} dt = \int \frac{(1 + 1/t^2) \, dt}{(1/t^2 - 1 + t^2)} = \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{(t - 1/t)^2 + 1}$,

Put $t - \frac{1}{t} = z$

$\therefore \left(1 + \frac{1}{t^2}\right) dt = dz = \int \frac{dz}{z^2 + 1} = \tan^{-1}(z) + C$

$= \tan^{-1}\left(\frac{t^2 - 1}{t}\right) + C = \tan^{-1}\left(\frac{\tan^2 x - 1}{\tan x}\right) + C$

$= \tan^{-1}(-2 \cot 2x) + C$

Hence, (d) is the correct answer.

Question 7:

$$\text{Let } I=\int \frac{e^{x}}{e^{4x}+e^{2x}+1}\,dx,\ J=\int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\,dx.$$

Then, for an arbitrary constant c, the value of J − I equals to

[IIT JEE 2008]

Solution:

(a) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$ (b) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+C$

(c) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+C$ (d) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

$$J=\int \frac{e^{3x}}{1+e^{2x}+e^{4x}}\,dx$$

$$J-I=\int \frac{(e^{3x}-e^{x})}{1+e^{2x}+e^{4x}}\,dx=\int \frac{(u^2-1)}{1+u^2+u^4}\,du \qquad (u=e^{x})$$

$$=\int \frac{\left(1-\frac{1}{u^2}\right)du}{1+\frac{1}{u^2}+u^2}=\int \frac{\left(1-\frac{1}{u^2}\right)du}{\left(u+\frac{1}{u}\right)^2-1}=\int \frac{dt}{t^2-1} \qquad \left(t=u+\frac{1}{u}\right)$$

$$=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+C=\frac{1}{2}\log\left|\frac{u^2-u+1}{u^2+u+1}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right|+C$$

Hence, (c) is the correct answer.

Question 8:

$$\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[(\sec^{-1}\sqrt{1+x^2})^2+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]dx,$$

$(x>0)$ is equal to

(a) $e^{\tan^{-1}x}\cdot\tan^{-1}x+C$

(b) $\frac{e^{\tan^{-1}x}\cdot(\tan^{-1}x)^2}{2}+C$

(c) $e^{\tan^{-1}x}\cdot(\sec^{-1}(\sqrt{1+x^2}))^2+C$

(d) $e^{\tan^{-1}x}\cdot(\operatorname{cosec}^{-1}(\sqrt{1+x^2}))^2+C$

Solution:

Note that $\sec^{-1}\sqrt{1+x^2}=\tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x$,

For $x>0$

$$\Rightarrow \qquad I=\int \frac{e^{\tan^{-1}x}}{1+x^2}\{(\tan^{-1}x)^2+2\tan^{-1}x\}\,dx,$$

Put $\tan^{-1}x=t$

$$=\int e^{t}(t^2+2t)\,dt=e^{t}\cdot t^2=e^{\tan^{-1}x}(\tan^{-1}x)^2+C$$

Hence, (c) is the correct answer.

Question 9:

The value of $\int \frac{(x-1)}{(x+1)\sqrt{x^3+x^2+x}}\,dx$, is

(a) $2\tan^{-1}\sqrt{\frac{x+1}{x}}+C$ (b) $\tan^{-1}\sqrt{\frac{x^2+x+1}{x}}+C$

(c) $2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}}+C$ (d) None of these

Solution:

Let $I = \int \frac{(x-1)}{(x+1)\sqrt{x^3+x^2+x}}\,dx$

$= \int \frac{(x^2-1)}{(x+1)^2\sqrt{x^3+x^2+x}}\,dx$

$= \int \frac{x^2(1-1/x^2)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}\,dx$

$= \int \frac{x^2(1-1/x^2)}{x(x+2+1/x)\cdot x\sqrt{x+1+1/x}}\,dx$

Put $x+\frac{1}{x}=t$,

$\Rightarrow (1-1/x^2)\,dx = dt$

$= \int \frac{dt}{(t+2)\sqrt{t+1}}$, which reduces to $\int \frac{dx}{P\sqrt{Q}}$.

Let $t+1=z^2$

$\therefore \quad dt = 2zdz = \int \frac{2z\,dz}{(z^2+1)\sqrt{z^2}}$

$= 2\int \frac{dz}{z^2+1} = 2\tan^{-1}(z)+C$

$= 2\tan^{-1}(\sqrt{t+1})+C = 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}}+C$

Hence, (c) is the correct answer.

Question 10:

The value of $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$, is

(a) $2\sin^{-1}\sqrt{\frac{x-a}{b-a}}+C$ (b) $2\sin^{-1}\sqrt{\frac{x-b}{b-a}}+C$

(c) $\sin^{-1}\sqrt{\frac{x-a}{b-a}}+C$ (d) None of these

Solution:

Let $x = a\cos^2\theta + b\sin^2\theta$ in the given integral.

So that, $dx = a(2\cos\theta)(-\sin\theta) + b(2\sin\theta)(\cos\theta)\,d\theta$

$dx = 2(b-a)\sin\theta\cos\theta\,d\theta$

$\therefore \quad I = \int \frac{2(b-a)\sin\theta\cos\theta\,d\theta}{\sqrt{(a\cos^2\theta+b\sin^2\theta-a)(b-a\cos^2\theta-b\sin^2\theta)}}$

$$= 2(b-a)\int \frac{\sin\theta \cos\theta \, d\theta}{\sqrt{(b\sin^2\theta - a\sin^2\theta)(b\cos^2\theta - a\cos^2\theta)}}$$

$$= 2(b-a)\int \frac{\sin\theta \cos\theta \, d\theta}{(b-a)\sin\theta \cos\theta} = 2\int 1 d\theta$$

$$= 2\theta + C = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}} + C$$

Hence, (a) is the correct answer.