Theorem 3 If a function f is differentiable at a point c, then it is also continuous at that point.

Proof Since f is differentiable at c, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

Therefore $$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

or $$\lim_{x \to c} [f(x)] - \lim_{x \to c} [f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right] \cdot \lim_{x \to c} [(x - c)]$$

$$= f'(c) \cdot 0 = 0$$

or $$\lim_{x \to c} f(x) = f(c)$$

Hence f is continuous at $x = c$.

Corollary 1 Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x) = |x|$ is a continuous function. Consider the left hand limit

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal, $\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

5.3.1 *Derivatives of composite functions*

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f, where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3 + 12x^2 + 6x + 1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\,(2x + 1)^2$$

Now, observe that $\quad f(x) = (h \text{ o } g)\,(x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\,(2x + 1)^2 = 3(2x + 1)^2 \,.\, 2 = 3t^2 \,.\, 2 = \frac{dh}{dt} \cdot \frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 (Chain Rule) Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \text{ o } u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u, v and w; i.e.,

$f = (w \text{ o } u) \text{ o } v$. If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \text{o} u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \text{ o } u)\,(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly proceed as follows:

$$y = \sin (x^2) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2)$$

$$= \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2$$

Example 22 Find the derivative of tan $(2x + 3)$.

Solution Let $f(x) = \tan (2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \text{ o } u)(x) = v(u(x)) = v(2x + 3) = \tan (2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2 (2x + 3)$$

Example 23 Differentiate $\sin (\cos (x^2))$ with respect to x.

Solution The function $f(x) = \sin (\cos (x^2))$ is a composition $f(x) = (w \text{ o } v \text{ o } u)(x)$ of the three functions u, v and w, where $u(x) = x^2$, $v(t) = \cos t$ and $w(s) = \sin s$. Put $t = u(x) = x^2$ and $s = v(t) = \cos t$. Observe that $\frac{dw}{ds} = \cos s$, $\frac{ds}{dt} = -\sin t$ and $\frac{dt}{dx} = 2x$ exist for all real x. Hence by a generalisation of chain rule, we have

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} = (\cos s) \cdot (-\sin t) \cdot (2x) = -2x \sin x^2 \cdot \cos (\cos x^2)$$