Question -

Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$. **(1978, 4M)**

Solution -

$$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n}$$

$$= [{}^{2n}C_0 + ({}^{2n}C_1)x + ({}^{2n}C_2)x^2 + \ldots + ({}^{2n}C_{2n})x^{2n}]$$

$$\times\left[{}^{2n}C_0 - ({}^{2n}C_1)\frac{1}{x} + ({}^{2n}C_2)\frac{1}{x^2} + \ldots + ({}^{2n}C_{2n})\frac{1}{x^{2n}}\right]$$

Independent terms of x on RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

$$\text{LHS} = (1+x)^{2n}\left(\frac{x-1}{x}\right)^{2n} = \frac{1}{x^{2n}}(1-x^2)^{2n}$$

Independent term of x on the LHS $= (-1)^n \cdot {}^{2n}C_n$.