Question -

Prove that $\frac{3!}{2(n+3)} = \sum_{r=0}^{n} (-1)^r \left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$. **(1997C, 5M)**

Solution -

$$\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{{}^{r+3}C_r}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{n! \cdot 3!}{(n-r)!\,(r+3)!} = 3! \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} \frac{(-1)^r \cdot (n+3)!}{(n-r)!(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)} \cdot \sum_{r=0}^{n} (-1)^r \cdot {}^{n+3}C_{r+3}$$

$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)} \sum_{s=3}^{n+3} (-1)^s \cdot {}^{n+3}C_3$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)} \left(\sum_{s=`0}^{n+3} (-1)^s \cdot {}^{n+3}C_s \right.$$

$$\left. - {}^{n+3}C_0 + {}^{n+3}C_1 - {}^{n+3}C_2 \right)$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)} \left\{ 0 - 1 + (n+3) - \frac{(n+3)(n+2)}{2!} \right\}$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)} \cdot \frac{(n+2)(2-n-3)}{2} = \frac{3!}{2(n+3)}$$