Question -

For any positive integers m, n (with $n \geq m$),

If $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots + \binom{m}{m} = \binom{n+1}{m+1}$$

or

Prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$

(IIT JEE 2000, 6M)

Solution -

$$\text{Let } S = \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots + \binom{m}{m} = \binom{n+1}{m+1} \quad \ldots\text{(i)}$$

It is obvious that, $n \geq m$. [given]

NOTE This question is based upon additive loop.

$$\text{Now, } S = \binom{m}{m} + \binom{m+1}{m} + \binom{m+2}{m} + \ldots + \binom{n}{m}$$

$$= \left\{\binom{m+1}{m+1} + \binom{m+1}{m}\right\} + \binom{m+2}{m} + \ldots \binom{n}{m}$$

$$\left[\because \binom{m}{m} = 1 = \binom{m+1}{m+1}\right]$$

$$= \binom{m+2}{m+1} + \binom{m+2}{m} + \ldots + \binom{n}{m}$$

$$[\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}]$$

$$= \binom{m+3}{m+1} + \ldots + \binom{n}{m}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \binom{n}{m+1} + \binom{n}{m} = \binom{n+1}{m+1}, \text{ which is true.} \quad \ldots\text{(ii)}$$

Again, we have to prove that

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\ldots+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$$

Let $S_1=\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\ldots+(n-m+1)\binom{m}{m}$

$$\left.\begin{aligned}=\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{m}\\ +\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{m}\\ +\binom{n-2}{m}+\ldots+\binom{m}{m}\\ \ldots\ldots\\ +\binom{m}{m}\end{aligned}\right\} n-m+1 \text{ rows}$$

Now, sum of the first row is $\binom{n+1}{m+1}$.

Sum of the second row is $\binom{n}{m+1}$.

Sum of the third row is $\binom{n-1}{m+1}$,

....

Sum of the last row is $\binom{m}{m}=\binom{m+1}{m+1}$.

Thus, $$S=\binom{n+1}{m+1}+\binom{n}{m+1}+\binom{n-1}{m+1}+\ldots+\binom{m+1}{m+1}=\binom{n+1+1}{m+2}=\binom{n+2}{m+2}$$

[from Eq. (i) replacing n by $n+1$ and m by $m+1$]