Question -

Prove that

$$2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n}{2}\binom{n-2}{k-2}-\ldots$$
$$+\ (-1)^k\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}$$ **(2003, 4 M)**

Solution -

To show that

$$2^{k}\cdot{}^{n}C_0\cdot{}^{n}C_k-2^{k-1}\cdot{}^{n}C_1\cdot{}^{n-1}C_{k-1}$$
$$+\,2^{k-2}\cdot{}^{n}C_2\cdot{}^{n-2}C_{k-2}-\ldots+\ (-1)^k\ {}^{n}C_k\,{}^{n-k}C_0={}^{n}C_k$$

Taking LHS

$$2^{k}\cdot{}^{n}C_0\cdot{}^{n}C_k-2^{k-1}\cdot{}^{n}C_1\cdot{}^{n-1}C_{k-1}\ +\ldots+(-1)^k\cdot{}^{n}C_k\cdot{}^{n-k}C_0$$

$$=\sum_{r=0}^{k}(-1)^r\cdot 2^{k-r}\cdot{}^{n}C_r\cdot{}^{n-r}C_{k-r}$$

$$=\sum_{r=0}^{k}(-1)^r\,2^{k-r}\cdot\frac{n!}{r!(n-r)!}\cdot\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$=\sum_{r=0}^{k}(-1)^r\cdot 2^{k-r}\cdot\frac{n!}{(n-k)!\cdot k!}\cdot\frac{k!}{r!(k-r)!}$$

$$=\sum_{r=0}^{k}(-1)^r\cdot 2^{k-r}\ {}^{n}C_k\cdot{}^{k}C_r\ =2^{k}\cdot{}^{n}C_k\left\{\sum_{r=0}^{k}(-1)^r\cdot\frac{1}{2^r}\cdot{}^{k}C_r\right\}$$

$$=2^{k}\cdot{}^{n}C_k\left(1-\frac{1}{2}\right)^k={}^{n}C_k=\text{RHS}$$