Question -

Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \dots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r$, $r \in \{1, 2, \dots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430} X$ is .......... . **(2018 Adv.)**

Solution -

We have,

$$X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \dots + 10\,({}^{10}C_{10})^2$$

$$\Rightarrow \quad X = \sum_{r=1}^{10} r({}^{10}C_r)^2 \Rightarrow X = \sum_{r=1}^{10} r\ {}^{10}C_r\ {}^{10}C_r$$

$$\Rightarrow \quad X = \sum_{r=1}^{10} r \times \frac{10}{r}\ {}^{9}C_{r-1}\ {}^{10}C_r \qquad \left[\because {}^{n}C_r = \frac{n}{r}\ {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow X = 10 \sum_{r=1}^{10} {}^{9}C_{r-1}\ {}^{10}C_r$$

$$\Rightarrow X = 10 \sum_{r=1}^{10} {}^{9}C_{r-1}\ {}^{10}C_{10-r} \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$$

$$\Rightarrow X = 10 \times {}^{19}C_9 \qquad [\because {}^{n-1}C_{r-1}\,{}^{n}C_{n-r} = {}^{2n-1}C_{n-1}]$$

$$\text{Now, } \frac{1}{1430} X = \frac{10 \times {}^{19}C_9}{1430} = \frac{{}^{19}C_9}{143} = \frac{{}^{19}C_9}{11 \times 13}$$

$$= \frac{19 \times 17 \times 16}{8} = 19 \times 34 = 646$$