Question -

If n is a positive integer and

$$(1 + x + x^2)^n = a_0 + a_1 x + \ldots + a_{2n} x^{2n}.$$

Then, show that, $a_0^2 - a_1^2 + \ldots + a_{2n}^2 = a_n$. **(1994, 5M)**

Solution -

$$(1 + x + x^2)^n = a_0 + a_1 x + \ldots + a_{2n} x^{2n} \qquad \ldots(i)$$

Replacing x by $-1/x$, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \ldots + \frac{a_{2n}}{x^{2n}} \qquad \ldots(ii)$$

Now, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_{2n}^2 =$ coefficient of the term independent of x in

$$[a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}] \times \left[a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}\right]$$

= Coefficient of the term independent of x in

$$(1 + x + x^2)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n$$

Now, RHS $= (1 + x + x^2)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n$

$$= \frac{(1 + x + x^2)^n \, (x^2 - x + 1)^n}{x^{2n}} = \frac{[(x^2 + 1)^2 - x^2]^n}{x^{2n}}$$

$$= \frac{(1 + 2x^2 + x^4 - x^2)^n}{x^{2n}} = \frac{(1 + x^2 + x^4)^n}{x^{2n}}$$

Thus, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_{2n}^2$

= Coefficient of the term independent of x in

$$\frac{1}{x^{2n}} (1 + x^2 + x^4)^n$$

= Coefficient of x^{2n} in $(1 + x^2 + x^4)^n$

= Coefficient of t^n in $(1 + t + t^2)^n = a_n$