Properties of binomial coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_r x^r + \ldots\ldots\ldots\ldots + C_nx^n$ ......(1)

where C_r denotes ${}^nC_r$

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n \qquad \ldots\ldots(2)$$

or $$\sum_{r=0}^{n} {}^nC_r = 2^n$$

(2) Again putting x = –1 in (1), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots\ldots + (-1)^n\, {}^nC_n = 0 \qquad \ldots\ldots(3)$$

or $$\sum_{r=0}^{n} (-1)^r\ {}^nC_r = 0$$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

from (2) and (3)

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots\ldots\ldots = 2^{n-1}$$

(4) Sum of two consecutive binomial coefficients

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad \Rightarrow \quad \text{L.H.S.} = {}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] = \frac{n!}{(n-r)!\,(r-1)!}\ \frac{(n+1)}{r(n-r+1)} = \frac{(n+1)!}{(n-r+1)!\,r!} = {}^{n+1}C_r = \text{R.H.S.}$$