5. Let 'd' be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$. ***(1995 - 5 Marks)***

Solution: -

5. Equation to the tangent at the point $P(a\cos\theta, b\sin\theta)$ on $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad ...(1)$$

$\therefore$ d = perpendicular distance of (1) from the centre (0, 0) of the ellipse

$$= \frac{1}{\sqrt{\frac{1}{a^2}\cos^2\theta + \frac{1}{b^2}\sin^2\theta}} = \frac{(ab)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore \quad 4a^2\left(1 - \frac{b^2}{d^2}\right) = 4a^2\left\{1 - \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2}\right\}$$

$$= 4(a^2 - b^2)\cos^2\theta = 4a^2e^2\cos^2\theta \quad ....(2)$$

The coordinates of focii F_1 and F_2 are

$F_1 = (ae, 0)$ and $F_2 = (-ae, 0)$

$$\therefore \quad PF_1 = \sqrt{[(a\cos\theta - ae)^2 + (b\sin\theta)^2]}$$

$$= \sqrt{[(a^2(\cos\theta - e)^2 + (b\sin\theta)^2]}$$

$$= \sqrt{[(a^2(\cos\theta - e)^2 + a^2(1-e^2)\sin^2\theta)]}$$

[Using $b^2 = a^2(1 - e^2)$]

$$= a\sqrt{[1 + e^2(1 - \sin^2\theta) - 2e\cos\theta]}$$

$$= a(1 - e\cos\theta)$$

Similarly, $PF_2 = a(1 + e\cos\theta)$

$$\therefore \quad (PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta \quad ...(3)$$

Hence from (2) and (3), we have

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{a^2}\right)$$