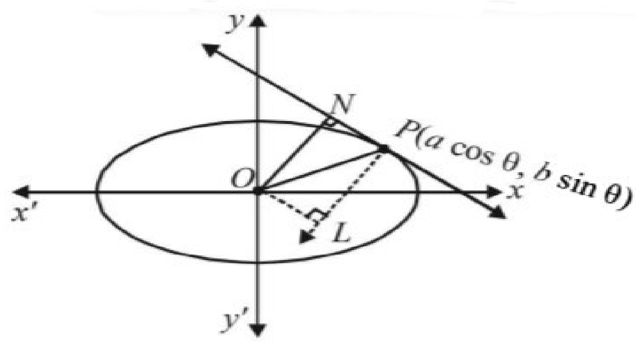
2. Find the co-ordinates of all the points P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P. ***(1999 - 10 Marks)***

Solution: -

2. The ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Since this ellipse is symmetrical in all four quadrants, either there exists no such P or four points, one in each qudrant. Without loss of generality we can assume that $a > b$ and P lies in first quadrant.



Let P ($a\cos\theta$, $b\sin\theta$) then equation of tangent is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$$\therefore \quad ON=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Equation of ON is, $\frac{x}{b}\sin\theta-\frac{y}{a}\cos\theta=0$

Equation of normal at P is

$$ax\sec\theta-\text{by}\ \text{cosec}\,\theta = a^2-b^2$$

$$\therefore OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\,\text{cosec}^2\theta}}=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2+b^2\cos^2\theta}}$$

and $NP=OL$

$$\therefore \quad NP = \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\therefore \quad Z = \text{Area of } OPN = \frac{1}{2} \times ON \times NP$$

$$= \frac{1}{2}ab(a^2 - b^2)\,\frac{\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

Let $u = \dfrac{a^2\sin^2\theta + b^2\cos^2\theta}{\sin\theta\cos\theta} = a^2\tan\theta + b^2\cot\theta$

$$\frac{du}{d\theta} = a^2\sec^2\theta - b^2\operatorname{cosec}^2\theta = 0 \Rightarrow \tan\theta = b/a$$

$\left(\dfrac{d^2u}{d\theta^2}\right)_{\tan^{-1} b/a} > 0$, u is minimum at $\theta = \tan^{-1} b/a$

So Z is maximum at $\theta = \tan^{-1} b/a$

$$\therefore \quad P \text{ is } \left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}}\right)$$

By symmetry, we have four such points

$$\left(\pm\frac{a^2}{\sqrt{a^2 + b^2}}, \pm\frac{b^2}{\sqrt{a^2 + b^2}}\right)$$