1. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,(a>b)$ meets the ellipse respectively, at P, Q, R. so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. ***(2000 - 7 Marks)***

Solution: -

1. Let A, B, C be the point on circle whose coordinates are
$A = [a\cos\theta, a\sin\theta]$

$$B=\left[a\cos\left(\theta+\frac{2\pi}{3}\right),a\sin\left(\theta+\frac{2\pi}{3}\right)\right]$$

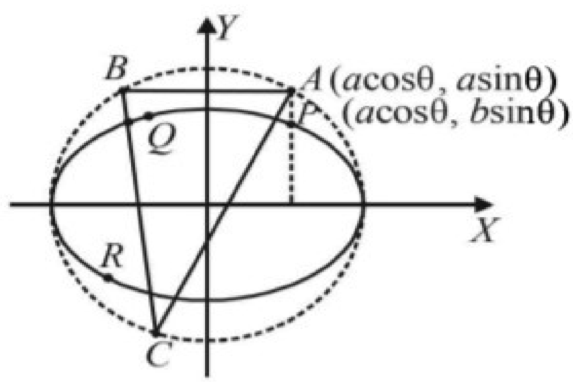
and $C=\left[a\cos\left(\theta+\frac{4\pi}{3}\right),a\sin\left(\theta+\frac{4\pi}{3}\right)\right]$

Further, $P[a\cos\theta, b\sin\theta]$ (Given)

$$Q=\left[a\cos\left(\theta+\frac{2\pi}{3}\right),b\sin\left(\theta+\frac{2\pi}{3}\right)\right]$$

and $R=\left[a\cos\left(\theta+\frac{4\pi}{3}\right),b\sin\left(\theta+\frac{4\pi}{3}\right)\right]$

It is given that P, Q, R are on the same side of x-axis as A, B, C.



So required normals to the ellipse are
$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$ ...(1)

$$ax\sec\left(\theta+\frac{2\pi}{3}\right)-by\,\mathrm{cosec}\left(\theta+\frac{2\pi}{3}\right)=a^2-b^2 \quad ...(2)$$

$$ax\sec\left(\theta+\frac{4\pi}{3}\right)-by\,\mathrm{cosec}\left(\theta+\frac{4\pi}{3}\right)=a^2-b^2 \quad ...(3)$$

Now, above three normals are concurrent

$\Rightarrow \quad \Delta=0$

$$\text{where } \Delta=\begin{vmatrix} \sec\theta & \cos\mathrm{ec}\theta & 1 \\ \sec\left(\theta+\frac{2\pi}{3}\right) & \mathrm{cosec}\left(\theta+\frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta+\frac{4\pi}{3}\right) & \mathrm{cosec}\left(\theta+\frac{4\pi}{3}\right) & 1 \end{vmatrix}$$

Multiplying and dividing the different rows R_1, R_2 and R_3 by $\sin\theta\cos\theta$,

$$\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{2\pi}{3}\right)$$

and $\sin\left(\theta+\frac{4\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)$ respectively, we get

$$\Delta=\frac{1}{\sin\theta\cos\theta\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{4\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)}\times$$

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}=0$$

[Operating $R_2 \to R_2+R_3$ and simplyfing R_2 we get $R_2 \equiv R_1$]