Tips & Tricks for Inverse Trigonometry

- While dealing with trigonometric or inverse trigonometric questions, one of the most important things is to know about proper domain & range, else you may choose wrong result even after solving correctly, so focus on that.

- Try to get a good knowledge of graphs of functions (v. imp). They will not only help in solving problems of this chapter, but also will be very useful in future chapters like Application of Derivatives, calculus etc.

- Interconversion, defining properly are v. imp. (from one trigonometric fun to other)

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$

(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$

(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$

(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & \text{if } x < -1 \end{cases}$

(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

Above set of definitions are for your reference, Try to prove them & try to draw their graphs. (will be very good) vimp

When you are seeing things like $2x\sqrt{1-x^2}$ the first thing you should think of is substitute x as $\cos\theta$ or $\sin\theta$ to be able to use $2\cos\theta\sin\theta = \sin 2\theta$ formulae (we want to use familiar trigonometric formulae.

$2x^2-1 \longrightarrow$ think of substituting x as $\cos\theta$
$(2\cos^2\theta - 1 = \cos 2\theta)$

$\dfrac{2x}{1-x^2}$ or $\dfrac{1-x^2}{1+x^2}$ or $\dfrac{2x}{1+x^2} \longrightarrow$ think of substituting x as $\tan\theta$.

So, taking these as hints, do try to define

1) $\sin^{-1}(2x\sqrt{1-x^2})$ in terms of $\sin^{-1}x$ or $\cos^{-1}x$

2) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ in terms of $\tan^{-1}x$ etc.

also draw graphs.

* In JEE, $\tan^{-1}x \pm \tan^{-1}(y)$, $\sin^{-1}(x) \pm \sin^{-1}(y)$, $\cos^{-1}(x) \pm \cos^{-1}(y)$ formulae have been traditionally very important.
Whenever you are asked summation of $\tan^{-1}$ of something you should try to break it as $\tan^{-1}(x) - \tan^{-1}(y)$

Eg $\displaystyle\sum_{n=1}^{n} \tan^{-1}\left(\frac{2}{n^2}\right)$

$$\tan^{-1}\left(\frac{2}{n^2}\right) = \tan^{-1}\left(\frac{2}{n^2-1+1}\right) = \tan^{-1}\left(\frac{2}{1+(n+1)(n-1)}\right)$$

$$= \tan^{-1}\left(\frac{(n+1)-(n-1)}{1+(n+1)(n-1)}\right)$$

$\left(\text{of the form } \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)\right)$

$$\therefore \tan^{-1}\left(\frac{2}{n^2}\right) = \tan^{-1}\left(\frac{(n+1)-(n-1)}{1+(n+1)(n-1)}\right) = \tan^{-1}(n+1) - \tan^{-1}(n-1)$$

Sum upto n terms,

$(n=1)$ $\quad \left[\cancel{\tan^{-1}(2)} - \tan^{-1}(1)\right] +$

$(n=2)$ $\quad \cancel{\tan^{-1}(3)} - \cancel{\tan^{-1}(2)}$

$\vdots$

$\left[\tan^{-1}(n+1) - \cancel{\tan^{-1}(n-1)}\right]$

So $\boxed{S_n = \tan^{-1}(n+1) - \pi/4}$

$$S_\infty = \lim_{n\to\infty} S_n$$

Questions for Practice

Find S_n & S_∞

1. $T_n = \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$
2. $T_n = \tan^{-1}\left(\frac{2n}{n^4+n^2+2}\right)$

- Whenever you see $\pm\sqrt{a^2-x^2}$ think of substituting $x = a\sin\theta / a\cos\theta$.

$x^2+a^2 \,/\, \sqrt{x^2+a^2} \rightarrow x = a\tan\theta / a\cot\theta$

$\sqrt{x^2-a^2} \longrightarrow x = a\sec\theta / a\,\mathrm{cosec}\,\theta$

* If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi$ then $x+y+z = xyz$

If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \frac{\pi}{2}$ then $xy + yz + zx = 1$

$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$