0 to 6.48 : coplanar lines+sums

Show that the lines x+3/-3=y-1/1=z-5/5;x+1/-1=y-2/2=z-5/5 are coplanar. Also find the equation of the plane containing the lines

$$\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{5} \quad \ldots (i)$$

and $$\frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5} \quad \ldots (ii)$$

These lines will be coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

$$\therefore \quad \begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}=2(5-10)-1(-15+5)=0$$

Hence lines are co-planar.

The equation of the plane containing two lines is

$$\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}=0$$

$\Rightarrow \quad -5(x+3)+10(y-1)-5(z-5)=0$

$\Rightarrow \quad -5x-15+10y-10-5z+25=0$

$\Rightarrow \quad -5x+10y-5z+0=0$

$\Rightarrow \quad -x+2y-z=0 \qquad$ or $\qquad x-2y+z=0$

6:48 to 14.02 : angle in terms of vector form

Find the angle between two vectors 5i – j + k and i + j – k

$\vec{a}$ = 5i – j + k and

$\vec{b}$ = i + j – k

The dot product is defined as

$\vec{a}.\vec{b}$= (5i – j + k)(i + j – k)

$\vec{a}.\vec{b}$= (5)(1) + (-1)(1) + (1)(-1)

$\vec{a}.\vec{b}$= 5-1-1

$\vec{a}.\vec{b}$= 3

The Magnitude of vectors is given by

$$|\vec{a}| = \sqrt{(5^2 + (-1)^2 + 1^2)} = \sqrt{27} = 5.19$$

$$|\vec{b}| = \sqrt{(1^2 + 1^2 + (-1)^2)} = \sqrt{3} = 1.73$$

The angle between the two vectors is

$$\theta = cos^{-1}\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\theta = cos^{-1}\frac{3}{(5.19)(1.73)}$$

$$\theta = cos^{-1}\frac{3}{8.97}$$

$$\theta = cos^{-1}(0.334)$$

$$\theta = 70.48^{\circ}$$

14.02 to 16.00 : angle in terms of cartesian form

(x – 1) / 2 = (y – 2) / 1 = (z – 3) / 2 and (x – 2) / 2 = (y – 1) / 2 = (z – 3) / 1 are the two lines in 3D space then the angle ∅ between them is given by

$m1 = 2i + j + 2k$

$|m1| = \sqrt{(2^2 + 1^2 + 2^2)} = \sqrt{9} = 3$

$m_2 = 2i + 2j + k$

$|m_2| = \sqrt{(2^2 + 2^2 + 1^2)} = \sqrt{9} = 3$

$\emptyset = \cos^{-1}\{(2\times2 + 1\times2 + 2\times1) / (3 \times 3)\}$

$\emptyset = \cos^{-1}\{(4 + 2 + 2) / 9\}$

$\emptyset = \cos^{-1}(8 / 9)$

16.00 to 21.25 : angle btw line and a plane

Example: Find the angle between the line and plane given below:

Line: $\frac{x-4}{2}, \frac{y+2}{6}, \frac{z-6}{-3}$

Plane: x – 2y + 3z + 4 = 0

Solution: Consider θ to be the angle between the line and the normal to the plane.

From the equation of the line, we find the direction vector as $\vec{d} = (2, 6, -3)$

From the equation of the plane, we find the normal vector as $\vec{n} = (1, -2, 3)$

Now we can use the equation discussed before to find the angle between the line and the plane:

$$\sin\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\sin\theta = \left|\frac{2\cdot1+6\cdot(-2)+(-3)\cdot3}{\sqrt{2^2+6^2+(-3)^2}\cdot\sqrt{1^2+(-2)^2+3^2}}\right| = \frac{19}{7\sqrt{14}}$$

Therefore, we can find the angle as: $\theta = \sin^{-1}\frac{19}{7\sqrt{14}}$

36.30 to 49.01 : distances and examples

Find the shortest distance between the lines l_1 and l_2 whose vector equations are:

$\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ (1)

and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$ (2)

Answer:-

Comparing (1) and (2) with $\vec{r_1} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r_2} = \vec{a_2} + \mu \vec{b_2}$, respectively, we get

$\vec{a_1} = \hat{i} + \hat{j}$, $\vec{b_1} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{a_2} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k}$

Therefore $(\vec{a_2} - \vec{a_1}) = (\hat{i} - \hat{k})$

and $\vec{b_1} \times \vec{b_2} = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

$$\left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{9+1+49} = \sqrt{59}.$$

Hence the shortest distance between the lines is given by

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{\left|\vec{b_1} \times \vec{b_2}\right|}\right| = \left|\frac{3 - 0 + 7}{\sqrt{59}}\right| = \frac{10}{\sqrt{59}}$$

49.01 to 54.28 : distance between parallel planes

Distance between two parallel planes 2x + y + 2z = 8 and 4x + 2y + 4z + 5 = 0 is

We know that the distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Therefore, the distance between $4x + 2y + 4z - 16 = 0$ and $4x + 2y + 4z + 5 = 0$ is

$$\left|\frac{5 + 16}{\sqrt{16 + 4 + 16}}\right| = \left|\frac{21}{\sqrt{36}}\right|$$

$$= \frac{21}{6} = \frac{7}{2}$$