11076CH03

Chapter 3

TRIGONOMETRIC FUNCTIONS

❖*A mathematician knows how to solve a problem, he can not solve it. – MILNE* ❖

3.1 Introduction

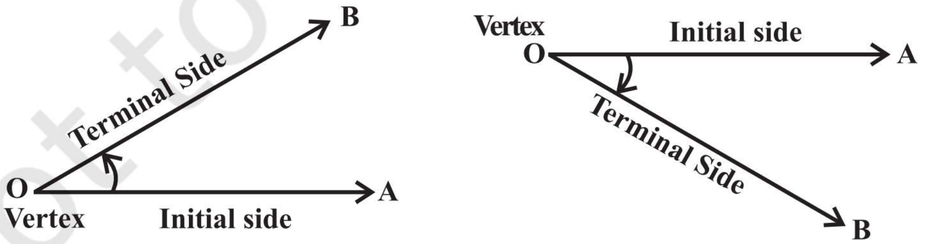
Arya Bhatt
(476-550)

The word 'trigonometry' is derived from the Greek words '*trigon*' and '*metron*' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this Chapter, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties.

3.2 Angles

Angle is a measure of rotation of a given ray about its initial point. The original ray is



(i)Positive angle Fig 3.1 (ii) Negative angle called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotation is called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative* (Fig 3.1).

The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. *one complete revolution* from the position of the initial side as indicated in Fig 3.2.

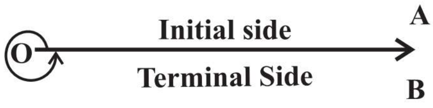


Fig 3.2

This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure.

3.2.1 *Degree measure*

If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one *degree*, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds . One sixtieth of a degree is called a *minute*, written as 1′, and one sixtieth of a minute is called a *second*, written as 1″. Thus, $1° = 60′$, $1′ = 60″$

Some of the angles whose measures are 360°,180°, 270°, 420°, – 30°, – 420° are shown in Fig 3.3.

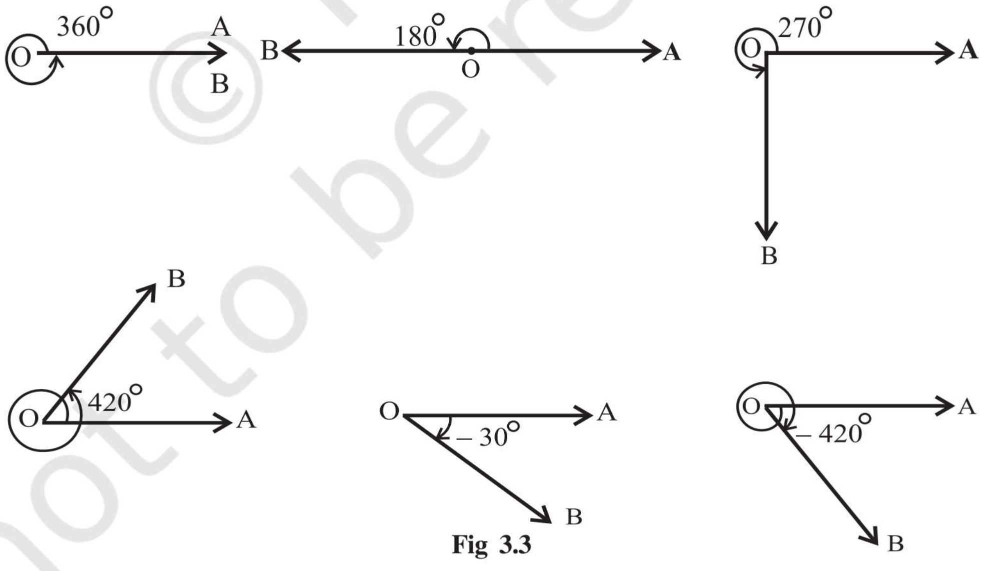


Fig 3.3

3.2.2 *Radian measure* There is another unit for measurement of an angle, called the *radian* measure. Angle subtended at the centre by an arc of length 1 unit in a unit circle (circle of radius 1 unit) is said to have a measure of 1 radian. In the Fig 3.4(i) to (iv), OA is the initial side and OB is the terminal side. The figures show the angles whose measures are 1 radian, –1 radian, $1\frac{1}{2}$ radian and $-1\frac{1}{2}$ radian.

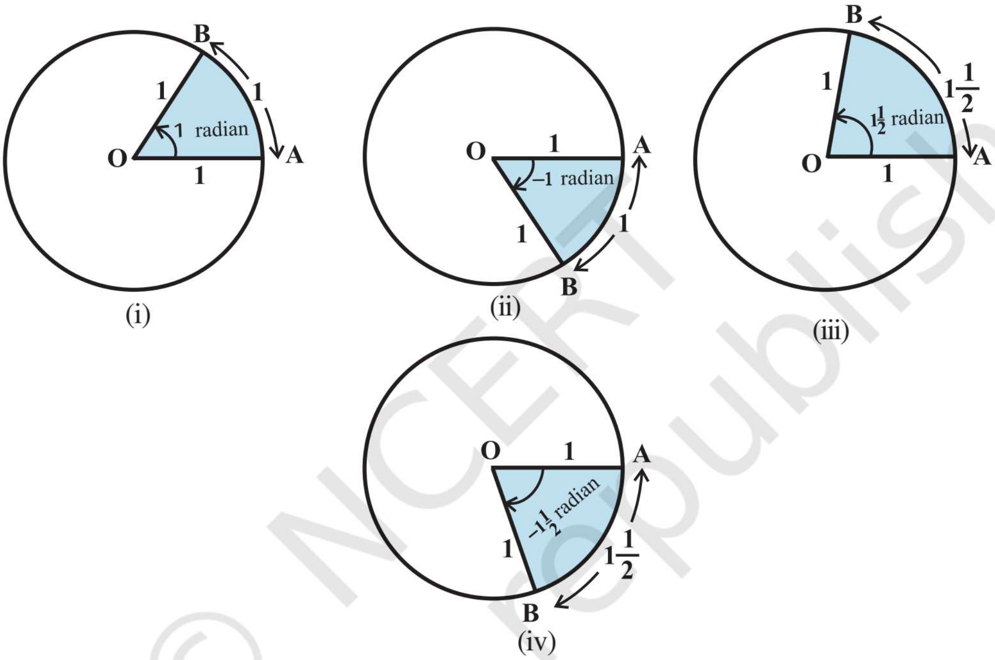


Fig 3.4 (i) to (iv)

We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length l will subtend an angle whose measure is $\frac{l}{r}$ radian. Thus, if in a circle of radius r, an arc of length l subtends an angle θ radian at the centre, we have

$$\theta = \frac{l}{r} \text{ or } l = r\,\theta.$$

3.2.3 *Relation between radian and real numbers*

Consider the unit circle with centre O. Let A be any point on the circle. Consider OA as initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle. Consider the line PAQ which is tangent to the circle at A. Let the point A represent the real number zero, AP represents positive real number and AQ represents negative real numbers (Fig 3.5). If we rope the line AP in the anticlockwise direction along the circle, and AQ in the clockwise direction, then every real number will correspond to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

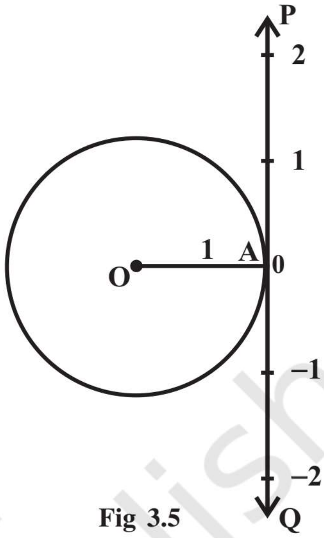


Fig 3.5

3.2.4 *Relation between degree and radian*

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that

$$2\pi \text{ radian} = 360° \quad \text{or} \quad \pi \text{ radian} = 180°$$

The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as $\frac{22}{7}$, we have

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\ 16' \text{ approximately.}$$

Also $$1° = \frac{\pi}{180} \text{ radian } = 0.01746 \text{ radian approximately.}$$

The relation between degree measures and radian measure of some common angles are given in the following table:

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Notational Convention

Since angles are measured either in degrees or in radians, we adopt the convention that whenever we write angle $\theta°$, we mean the angle whose degree measure is θ and whenever we write angle β, we mean the angle whose radian measure is β.

Note that when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, $\pi = 180°$ and $\frac{\pi}{4} = 45°$ are written with the understanding that π and $\frac{\pi}{4}$ are radian measures. Thus, we can say that

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Example 1 Convert $40°\ 20'$ into radian measure.

Solution We know that $180° = \pi$ radian.

Hence $\quad 40°\ 20' = 40\frac{1}{3}$ degree $= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $\quad 40°\ 20' = \frac{121\pi}{540}$ radian.

Example 2 Convert 6 radians into degree measure.

Solution We know that π radian = 180°.

Hence $\quad 6$ radians $= \frac{180}{\pi} \times 6$ degree $= \frac{1080 \times 7}{22}$ degree

$= 343\frac{7}{11}$ degree $\quad = 343° + \frac{7 \times 60}{11}$ minute $\quad$ [as 1° = 60′]

$= 343° + 38' + \frac{2}{11}$ minute $\quad$ [as 1′ = 60″]

$= 343° + 38' + 10.9'' \quad = 343°38'\ 11''$ approximately.

Hence $\quad$ 6 radians = $343°\ 38'\ 11''$ approximately.

Example 3 Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).

Solution Here $l = 37.4$ cm and $\theta = 60° = \frac{60\pi}{180}$ radian $= \frac{\pi}{3}$

Hence, by $r = \frac{l}{\theta}$, we have

$$r = \frac{37.4 \times 3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7 \text{ cm}$$

Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$).

Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}.$$

Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii.

Solution Let r_1 and r_2 be the radii of the two circles. Given that

$$\theta_1 = 65° = \frac{\pi}{180} \times 65 = \frac{13\pi}{36} \text{ radian}$$

and $$\theta_2 = 110° = \frac{\pi}{180} \times 110 = \frac{22\pi}{36} \text{ radian}$$

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$$\frac{13\pi}{36} \times r_1 = \frac{22\pi}{36} \times r_2, \text{ i.e., } \frac{r_1}{r_2} = \frac{22}{13}$$

Hence $r_1 : r_2 = 22 : 13$.

EXERCISE 3.1

1. Find the radian measures corresponding to the following degree measures:

(i) 25° (ii) – 47°30′ (iii) 240° (iv) 520°

2. Find the degree measures corresponding to the following radian measures (Use $\pi = \frac{22}{7}$).

(i) $\frac{11}{16}$ (ii) -4 (iii) $\frac{5\pi}{3}$ (iv) $\frac{7\pi}{6}$

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?
4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = \frac{22}{7}$).
5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.
6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.
7. Find the angle in radian through which a pendulum swings if its length is 75 cm and th e tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

3.3 Trigonometric Functions

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

Consider a unit circle with centre at origin of the coordinate axes. Let P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x (Fig 3.6).

We define $\cos x = a$ and $\sin x = b$ Since ΔOMP is a right triangle, we have

$OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$

Thus, for every point on the unit circle, we have

$a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$,

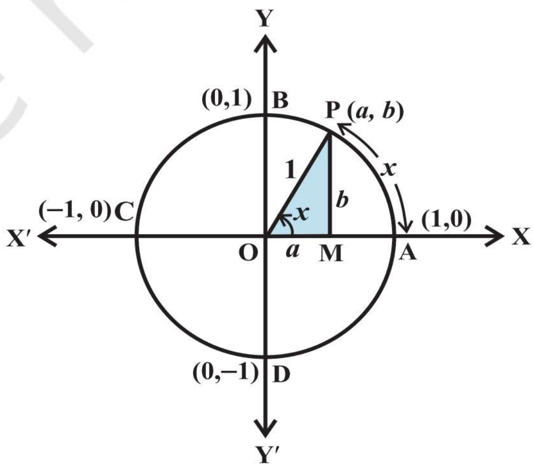


Fig 3.6

$\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$$\cos 0° = 1 \qquad \sin 0° = 0,$$

$$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1$$

$$\cos\pi = -1 \qquad \sin\pi = 0$$

$$\cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in \mathbf{Z}, \quad \cos (2n\pi + x) = \cos x,\ n \in \mathbf{Z}$$

Further, $\sin x = 0$, if $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

$\sin x = 0$ implies $x = n\pi$, where n is any integer

$\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$$\text{cosec } x = \frac{1}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \text{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$$\cos(-x) = \cos x$$

and $$\sin(-x) = -\sin x$$

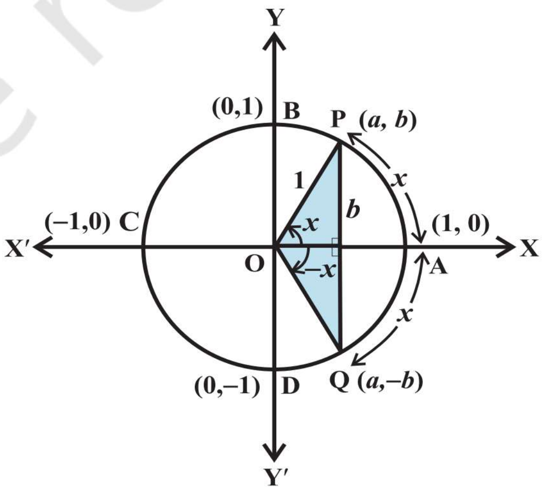


Fig 3.7

Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and

$-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\text{cosec } x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \leq \sin x \leq 1 \text{ and } -1 \leq \cos x \leq 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \leq y \leq 1$.