Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

4.1 Overview

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

4.1.1 *The principle of mathematical induction*

Let P(n) be a given statement involving the natural number n such that

(i) The statement is true for $n = 1$, i.e., P(1) is true (or true for any fixed natural number) and

(ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

4.2 Solved Examples

Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all $n \in \mathbf{N}$, that :

Example 1 $1 + 3 + 5 + ... + (2n - 1) = n^2$

Solution Let the given statement P(n) be defined as P(n) : $1 + 3 + 5 + ... + (2n - 1) = n^2$, for $n \in \mathbf{N}$. Note that P(1) is true, since

$$P(1) : 1 = 1^2$$

Assume that P(k) is true for some $k \in \mathbf{N}$, i.e.,

$$P(k) : 1 + 3 + 5 + ... + (2k - 1) = k^2$$

Now, to prove that P(k + 1) is true, we have

$$1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$$

$$= k^2 + (2k + 1) \qquad \text{(Why?)}$$

$$= k^2 + 2k + 1 = (k + 1)^2$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all $n \in$ **N**.

Example 2 $\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$, for all natural numbers $n \geq 2$.

Solution Let the given statement P(n), be given as

$$P(n): \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

We observe that

$$P(2): \sum_{t=1}^{2-1} t(t+1) = \sum_{t=1}^{1} t(t+1) = 1.2 = \frac{1.2.3}{3}$$

$$= \frac{2.(2-1)(2+1)}{3}$$

Thus, P(n) in true for $n = 2$.

Assume that P(n) is true for $n = k \in$ **N**.

i.e.,
$$P(k): \sum_{t=1}^{k-1} t(t+1) = \frac{k(k-1)(k+1)}{3}$$

To prove that P(k + 1) is true, we have

$$\sum_{t=1}^{(k+1-1)} t(t+1) = \sum_{t=1}^{k} t(t+1)$$

$$= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$= k(k+1)\left[\frac{k-1+3}{3}\right] = \frac{k(k+1)(k+2)}{3}$$

$$= \frac{(k+1)((k+1)-1))((k+1)+1)}{3}$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 3 $\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$, for all natural numbers, $n \geq 2$.

Solution Let the given statement be P(n), i.e.,

$$P(n):\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}, \text{ for all natural numbers, } n \geq 2$$

We, observe that P (2) is true, since

$$\left(1-\frac{1}{2^2}\right)=1-\frac{1}{4}=\frac{4-1}{4}=\frac{3}{4}=\frac{2+1}{2\times 2}$$

Assume that P(n) is true for some $k \in$ **N**, i.e.,

$$P(k): \quad 1-\frac{1}{2^2} \; . \; 1-\frac{1}{3^2} \; ... \; 1-\frac{1}{k^2}=\frac{k+1}{2k}$$

Now, to prove that P (k + 1) is true, we have

$$1-\frac{1}{2^2} \; . \; 1-\frac{1}{3^2} \; ... \; 1-\frac{1}{k^2} \; . \; 1-\frac{1}{(k+1)^2}$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)=\frac{k^2+2k}{2k\,(k+1)}=\frac{(k+1)+1}{2\,(k+1)}$$

Thus, P (k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers, $n \geq 2$.

Example 4 $2^{2n} - 1$ is divisible by 3.

Solution Let the statement P(n) given as

P(n) : $2^{2n} - 1$ is divisible by 3, for every natural number n.

We observe that P(1) is true, since

$$2^2 - 1 = 4 - 1 = 3.1 \text{ is divisible by } 3.$$

Assume that P(n) is true for some natural number k, i.e.,

P(k): $2^{2k} - 1$ is divisible by 3, i.e., $2^{2k} - 1 = 3q$, where $q \in$ **N**

Now, to prove that P(k + 1) is true, we have

$$P(k+1) : 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} . 2^2 - 1$$

$$= 2^{2k} . 4 - 1 = 3.2^{2k} + (2^{2k} - 1)$$

$$= 3.2^{2k} + 3q$$
$$= 3\,(2^{2k} + q) = 3m, \text{ where } m \in \mathbf{N}$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

Example 5 $2n + 1 < 2^n$, for all natual numbers $n \geq 3$.

Solution Let P(n) be the given statement, i.e., P(n) : $(2n + 1) < 2^n$ for all natural numbers, $n \geq 3$. We observe that P(3) is true, since

$$2.3 + 1 = 7 < 8 = 2^3$$

Assume that P(n) is true for some natural number k, i.e., $2k + 1 < 2^k$

To prove P(k + 1) is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have

$$2(k + 1) + 1 = 2\,k + 3$$
$$= 2k + 1 + 2 < 2^k + 2 < 2^k\,.\,2 = 2^{k+1}.$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers, $n \geq 3$.

Long Answer Type

Example 6 Define the sequence $a_1, a_2, a_3 ...$ as follows :

$a_1 = 2$, $a_n = 5\,a_{n-1}$, for all natural numbers $n \geq 2$.

(i) Write the first four terms of the sequence.

(ii) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2.5^{n-1}$ for all natural numbers.

Solution

(i) We have $a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$

$a_3 = 5a_{3-1} = 5a_2 = 5.10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$

(ii) Let P(n) be the statement, i.e.,

P(n) : $a_n = 2.5^{\,n-1}$ for all natural numbers. We observe that P(1) is true

Assume that P(n) is true for some natural number k, i.e., P(k) : $a_k = 2.5^{k-1}$.

Now to prove that P (k + 1) is true, we have

$$P(k+1) : a_{k+1} = 5.a_k = 5 \,.\, (2.5^{k-1})$$
$$= 2.5^k = 2.5^{(k+1)-1}$$

Thus P(k + 1) is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers.

Example 7 The distributive law from algebra says that for all real numbers c, a_1 and a_2, we have $c\,(a_1 + a_2) = ca_1 + ca_2$.

Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if $c, a_1, a_2, ..., a_n$ are any real numbers, then

$$c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... + ca_n$$

Solution Let P(n) be the given statement, i.e.,

P(n) : $c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... ca_n$ for all natural numbers $n \geq 2$, for $c, a_1, a_2, ... a_n \in \mathbf{R}$.

We observe that P(2) is true since

$$c(a_1 + a_2) = ca_1 + ca_2 \qquad \text{(by distributive law)}$$

Assume that P(n) is true for some natural number k, where $k > 2$, i.e.,

$$P(k) : c\,(a_1 + a_2 + ... + a_k) = ca_1 + ca_2 + ... + ca_k$$

Now to prove P(k + 1) is true, we have

$$P(k+1) : c\,(a_1 + a_2 + ... + a_k + a_{k+1})$$
$$= c\,((a_1 + a_2 + ... + a_k) + a_{k+1})$$
$$= c\,(a_1 + a_2 + ... + a_k) + ca_{k+1} \qquad \text{(by distributive law)}$$
$$= ca_1 + ca_2 + ... + ca_k + ca_{k+1}$$

Thus P(k + 1) is true, whenever P (k) is true.

Hence, by the principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 8 Prove by induction that for all natural number n

$$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta)$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Solution Consider P (n) : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that
P (1) is true, since

$$P(1) : \sin\alpha = \frac{\sin(\alpha + 0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that P(n) is true for some natural numbers k, i.e.,
P (k) : sin α + sin (α + β) + sin (α + 2β) + ... + sin (α + (k – 1)β)

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that P (k + 1) is true, we have
P (k + 1) : sin α + sin (α + β) + sin (α + 2β) + ... + sin (α + (k – 1) β) + sin (α + kβ)

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\,(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thus P (k + 1) is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural number n.

Example 9 Prove by the Principle of Mathematical Induction that
$1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers n.

Solution Let P(n) be the given statement, that is,

P(n) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers n.

Note that P (1) is true, since

$$P(1) : 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that P(n) is true for some natural number k, i.e.,

P(k) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! = (k + 1)! - 1$

To prove P (k + 1) is true, we have

$$\begin{aligned} P(k+1) : 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! + (k + 1) \times (k + 1)! \\ = (k + 1)! - 1 + (k + 1)! \times (k + 1) \\ = (k + 1 + 1)\ (k + 1)! - 1 \\ = (k + 2)\ (k + 1)! - 1 = ((k + 2)! - 1 \end{aligned}$$

Thus P (k + 1) is true, whenever P (k) is true. Therefore, by the Principle of Mathematical Induction, P (n) is true for all natural number n.

Example 10 Show by the Principle of Mathematical Induction that the sum S_n of the n term of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2$ ... is given by

$$S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Solution Here $P(n) : S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$

Also, note that any term T_n of the series is given by

$$T_n = \begin{cases} n^2 \text{ if } n \text{ is odd} \\ 2n^2 \text{ if } n \text{ is even} \end{cases}$$

We observe that P(1) is true since

$$P(1) : S_1 = 1^2 = 1 = \frac{1.2}{2} = \frac{1^2.(1+1)}{2}$$

Assume that P(k) is true for some natural number k, i.e.

Case 1 When k is odd, then $k + 1$ is even. We have

$$P(k+1) : S_{k+1} = 1^2 + 2 \times 2^2 + ... + k^2 + 2 \times (k+1)^2$$

$$= \frac{k^2(k+1)}{2} + 2 \times (k+1)^2$$

$$= \frac{(k+1)}{2}[k^2 + 4(k+1)] \text{ (as } k \text{ is odd, } 1^2 + 2 \times 2^2 + ... + k^2 = k^2 \frac{(k+1)}{2})$$

$$= \frac{k+1}{2}[k^2 + 4k + 4]$$

$$= \frac{k+1}{2}(k+2)^2 = (k+1)\frac{[(k+1)+1]^2}{2}$$

So P(k + 1) is true, whenever P(k) is true in the case when k is odd.

Case 2 When k is even, then $k + 1$ is odd.

Now, $P(k+1): 1^2 + 2 \times 2^2 + ... + 2.k^2 + (k+1)^2$

$$= \frac{k(k+1)^2}{2} + (k+1)^2 \quad \left(\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + ... + 2k^2 = k\ \frac{(k+1)^2}{2}\right)$$

$$= \frac{(k+1)^2 (k+2)}{2} = \frac{(k+1)^2 ((k+1)+1)}{2}$$

Therefore, P (k + 1) is true, whenever P (k) is true for the case when k is even. Thus P (k + 1) is true whenever P (k) is true for any natural numbers k. Hence, P (n) true for all natural numbers.

Objective Type Questions

Choose the correct answer in Examples 11 and 12 (M.C.Q.)

Example 11 Let P(n) : "$2^n < (1 \times 2 \times 3 \times ... \times n)$". Then the smallest positive integer for which P (n) is true is

(A) 1 (B) 2 (C) 3 (D) 4

Solution Answer is D, since

$P(1): 2 < 1$ is false

$P(2): 2^2 < 1 \times 2$ is false

$P(3): 2^3 < 1 \times 2 \times 3$ is false

But $P(4): 2^4 < 1 \times 2 \times 3 \times 4$ is true

Example 12 A student was asked to prove a statement P (n) by induction. He proved that P (k + 1) is true whenever P (k) is true for all $k > 5 \in \mathbf{N}$ and also that P (5) is true. On the basis of this he could conclude that P (n) is true

(A) for all $n \in \mathbf{N}$ (B) for all $n > 5$

(C) for all $n \geq 5$ (D) for all $n < 5$

Solution Answer is (C), since P(5) is true and P(k + 1) is true, whenever P (k) is true.

Fill in the blanks in Example 13 and 14.

Example 13 If P (n) : "$2.4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in \mathbf{N}$" is true, then the value of λ is _____

Solution Now, for $n = 1$,

$2.4^{2+1} + 3^{3+1} = 2.4^3 + 3^4 = 2.64 + 81 = 128 + 81 = 209,$

for $n = 2,\ 2.4^5 + 3^7 = 8.256 + 2187 = 2048 + 2187 = 4235$

Note that the H.C.F. of 209 and 4235 is 11. So $2.4^{2n+1} + 3^{3n+1}$ is divisible by 11. Hence, λ is 11

Example 14 If P (n) : "$49^n + 16^n + k$ is divisible by 64 for $n \in \mathbf{N}$" is true, then the least negative integral value of k is _______.

Solution For $n = 1$, P(1) : $65 + k$ is divisible by 64.

Thus k, should be – 1 since, 65 – 1 = 64 is divisible by 64.

Example 15 State whether the following proof (by mathematical induction) is true or false for the statement.

$$P(n):\ 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof By the Principle of Mathematical induction, P(n) is true for $n = 1$,

$1^2 = 1 = \dfrac{1(1+1)(2\cdot1+1)}{6}$. Again for some $k \geq 1$, $k^2 = \dfrac{k(k+1)(2k+1)}{6}$. Now we prove that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Solution False

Since in the inductive step both the inductive hypothesis and what is to be proved are wrong.

4.3 EXERCISE

Short Answer Type

1. Give an example of a statement P(n) which is true for all $n \geq 4$ but P(1), P(2) and P(3) are not true. Justify your answer.
2. Give an example of a statement P(n) which is true for all n. Justify your answer.

   Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :
3. $4^n - 1$ is divisible by 3, for each natural number n.
4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n.
5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.
6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

7. For any natural number n, $7^n - 2^n$ is divisible by 5.

8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

10. $n(n^2 + 5)$ is divisible by 6, for each natural number n.

11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.

12. $2n < (n + 2)!$ for all natural number n.

13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

14. $2 + 4 + 6 + \ldots + 2n = n^2 + n$ for all natural numbers n.

15. $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all natural numbers n.

16. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$ for all natural numbers n.

Long Answer Type

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence $a_1, a_2, a_3 \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.

18. A sequence $b_0, b_1, b_2 \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural number n using mathematical induction.

19. A sequence $d_1, d_2, d_3 \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in \mathbf{N}$.

20. Prove that for all $n \in \mathbf{N}$

$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n-1)\beta)$

$$= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

21. Prove that, $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$, for all $n \in \mathbf{N}$.

22. Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$, for all $n \in \mathbf{N}$.

23. Show that $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number for all $n \in$ **N.**

24. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n > 1$.

25. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in$ **N**.

Objective Type Questions

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in$ **N**, then the least positive integral value of k is

(A) 5 (B) 3 (C) 7 (D) 1

27. For all $n \in$ **N**, $3.5^{2n+1} + 2^{3n+1}$ is divisible by

(A) 19 (B) 17 (C) 23 (D) 25

28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(A) 1 (B) 2 (C) 3 (D) 4

Fill in the blanks in the following :

29. If P(n) : $2n < n!$, $n \in$ **N**, then P(n) is true for all $n \geq$ ________.

State whether the following statement is true or false. Justify.

30. Let P(n) be a statement and let P(k) ⇒ P(k + 1), for some natural number k, then P(n) is true for all $n \in$ **N**.

Hint: Read the defn of PMI